Abelian Categories and Mitchell's Embedding Theorem

James Bailie

June 16, 2017

1 Introduction

This report introduces abelian categories and develops some basic theory about these categories. We also sketch a proof of Mitchell's embedding theorem.

Abelian categories are particularly nice categories — they have direct sums, kernels, cokernels, exact sequences and the homsets have the structure of abelian groups. They are a categorification of abelian groups and are therefore useful in proving general theory in areas such as homological algebra, algebraic geometry, cohomology and elsewhere. Buchsbaum and Grothendieck independently developed abelian categories in 1955 [Bu] and 1957 [Gr] respectively.

Mitchell's embedding theorem [Mi] states that every small abelian category is equivalent to a full subcategory of R-**Mod** for some ring R. This allows one to think of an abstract abelian category as a concrete category of modules, which is useful since modules are well understood and, arguably, easier to work in. In particular, objects in the category of modules are actually sets; the notion of an element makes sense; and, therefore, we have useful proof techniques, such as element-wise diagram chasing.

Sections 2-4 introduce a weaker version of abelian categories, which are called additive categories. In section 5, we give the formal definition of an abelian category and provide some typical examples. Section 6 states Michell's embedding theorem and further discusses its uses, limitations, and criticisms. The subsequent sections provide a proof of this theorem, in the process of which we develop some theory of abelian groups. Section 9 is a proof of the snake lemma for abelian categories, by the standard diagram chase. Such a proof is only possible by Mitchell's embedding theorem and thus provides an important application of the theorem.

We write composition of morphisms f and g as fg, supressing the $\circ$ symbol.

2 Additive Categories

To define an abelian category, it is natural to first define a generalisation, which we call an additive category. These categories are interesting in their own right and we will develop some theory concerning them.

Definition 1. A (locally small) category $\mathscr{C}$ is an ***Ab**-category* (or *preadditive*) if all homsets $\mathscr{C}(X \to Y)$ are abelian groups (written additively) such that composition is biadditive: for morphisms

$$X \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y \underset{g_2}{\overset{g_1}{\rightrightarrows}} Z$$

we have $g_1(f_1 + f_2) = g_1 f_1 + g_1 f_2$ and $(g_1 + g_2) f_1 = g_1 f_1 + g_2 f_1$.

Alternatively, an **Ab**-category is a category enriched over **Ab**.

Recall that a one object category where all morphisms are invertible is exactly a group. In the same way, a one object **Ab**-category is exactly the same as a ring. The category R-**Mod** of left R-modules for any ring R is also an **Ab**-category.

In an **Ab**-category $\mathscr{C}$, the additive identity of any homset is denoted by 0.

Proposition 2. *An* **Ab***-category has the following ring-like properties:*

1. $X \xrightarrow{f} Y \xrightarrow{0} Z = X \xrightarrow{0} Z = X \xrightarrow{0} Y \xrightarrow{g} Z$,
2. $(-f)g = -(fg) = f(-g)$.

Proof. To prove 1., note $f0 = f(0+0) = f0 + f0$. The second equality follows similarly.

Using 1., we have $(-f)g + fg = (-f+f)g = 0g = 0$ which implies $(-f)g = -(fg)$. The second equality is proven in the same manner. □

Definition 3. An **Ab**-category $\mathscr{C}$ is *additive* if there is a zero object $0 \in \mathscr{C}$ and products $X \times Y$ exist for every pair $X, Y \in \mathscr{C}$.

Binary coproducts also exist in additive categories and are in fact equal to the corresponding binary product. This a corollary of the following theorem. We denote products $X \times Y$ by $X \oplus Y$.

Theorem 4. *In an additive category $\mathscr{C}$, there exist unique morphisms*

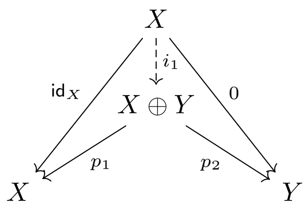
$$X \xrightarrow{i_1} X \oplus Y \xleftarrow{i_2} Y$$

such that $p_1 i_1 = \mathsf{id}_X$, $p_2 i_2 = \mathsf{id}_Y$, $p_1 i_2 = 0$, $p_2 i_1 = 0$ and $i_1 p_1 + i_2 p_2 = \mathsf{id}_{X \oplus Y}$, where

$$X \xleftarrow{p_1} X \oplus Y \xrightarrow{p_2} Y$$

are the projection maps of the product $X \oplus Y$.

Proof. First, we show existence. Define i_1 as the unique map between the cones $(X, \mathsf{id}_X, X \xrightarrow{0} Y)$ and $(X \oplus Y, X \oplus Y \xrightarrow{p_1} X, X \oplus Y \xrightarrow{p_2} Y)$:



Similarly define i_2. By the above diagram, $p_1 i_1 = \mathsf{id}_X$ and $p_2 i_1 = 0$ are true. The analogous diagram for i_2 shows $p_2 i_2 = \mathsf{id}_Y$ and $p_1 i_2 = 0$.

Then it follows that

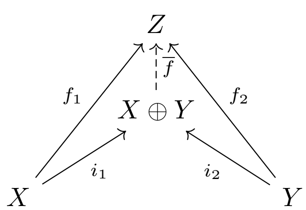
$$p_1(i_1 p_1 + i_2 p_2) = p_1 i_1 p_1 = p_1 \text{ and } p_2(i_1 p_1 + i_2 p_2) = p_2 i_2 p_2 = p_2.$$

Therefore, $i_1 p_1 + i_2 p_2$ is a map between the cone $(X \oplus Y, p_1, p_2)$ and itself. By uniqueness of such maps, $i_1 p_1 + i_2 p_2 = \mathsf{id}_{X \oplus Y}$.

Now we show uniqueness. Suppose $i_1' : X \to X \oplus Y$ satisfies $p_1 i_1' = \mathsf{id}_X$ and $p_2 i_1' = 0$. Then i_1' is a map between the cones $(X, \mathsf{id}_X, X \xrightarrow{0} Y)$ and $(X \oplus Y, X \oplus Y \xrightarrow{p_1} X, X \oplus Y \xrightarrow{p_2} Y)$. Thus, $i_1' = i_1$ by uniqueness of such maps. Uniqueness of i_2 is proven analogously. □

Corollary 5. *Suppose maps i_1, i_2 are constructed as above. Then $(X \oplus Y, i_1, i_2)$ is the coproduct of X and Y.*

Proof. Suppose we have maps $f_1 : X \to Z$ and $f_2 : Y \to Z$. Define $\overline{f} = f_1 p_1 + f_2 p_2$. We will show that $\overline{f}$ is the unique map such that



commutes. Firstly, $\overline{f}$ satisfies the commutativity relations:

$$\overline{f} i_1 = f_1 p_1 i_1 + f_2 p_2 i_1 = f_1 \text{ and } \overline{f} i_2 = f_1 p_1 i_2 + f_2 p_2 i_2 = f_2,$$

by Theorem 4.

Now, suppose $g : X \oplus Y \to Z$ satisfies the commutative diagram above. Then

$$g = g(i_1 p_1 + i_2 p_2) = g i_1 p_1 + g i_2 p_2 = f_1 p_1 + f_2 p_2 = \overline{f}.$$ □

Using this corollary, we get an equivalent definition of an additive category: an **Ab**-category $\mathscr{C}$ is additive if there is a zero object $0 \in \mathscr{C}$ and for every pair $X, Y \in \mathscr{C}$, there is a unique (up to unique isomorphism) object Z with maps

$$X \underset{p_1}{\overset{i_1}{\rightleftarrows}} Z \underset{p_2}{\overset{i_2}{\leftrightarrows}} Y$$

such that $p_1 i_1 = \mathsf{id}_X$, $p_2 i_2 = \mathsf{id}_Y$ and $i_1 p_1 + i_2 p_2 = \mathsf{id}_Z$.

In an additive category, $\mathscr{C}(0 \to 0)$ is the trivial group. The obvious examples of additive categories are **Vec** and **Ab**.

Definition 6. A functor $F : \mathscr{C} \to \mathscr{D}$ between additive categories $\mathscr{C}, \mathscr{D}$ is called *additive* if the induced function $\mathscr{C}(X \to Y) \longrightarrow \mathscr{D}(F(X) \to F(Y))$ is an abelian group homomorphism.

In section 7, we shall see that additive functors are the natural choice of 'nice' maps between additive categories.

Proposition 7. *If $F : \mathscr{C} \to \mathscr{D}$ is an additive functor then $F(0) \cong 0$.*

Proof. We know F preserves identities in the sense that $F(\mathsf{id}_X) = \mathsf{id}_{F(X)}$. Since F induces a homomorphisms on hom-sets, F also preserves additive identities in the sense that $F(X \xrightarrow{0} Y) = F(X) \xrightarrow{0} F(Y)$. Since $\mathsf{id}_0 = 0$, it follows $\mathsf{id}_{F(0)}$ is the zero map. So for any map $f : F(0) \to X$, we have $f = f\mathsf{id}_{F(0)} = f0 = 0$. Similarly, any map $g : X \to F(0)$ is also zero, proving that $F(0)$ is a zero object in $\mathscr{D}$. ☐

3 Kernels and Cokernels

In this section, we generalise the definition of kernels and cokernels from abstract algebra. We think of kernels as the part of the domain that maps to zero and cokernels as the dual - the part of the codomain not in the image. The following two definitions formalise these notions in the most general setting possible. Later in this section, we will specialise to additive categories.

Definition 8. Let $\mathscr{C}$ be a category with initial object 0 and $f : X \to Y$. Consider the pullback, if it exists, of f and the unique map $0 \to Y$:

$$\begin{array}{ccc} K & \longrightarrow & 0 \\ {\scriptstyle k}\downarrow & & \downarrow \\ X & \underset{f}{\longrightarrow} & Y. \end{array}$$

The *kernel* of f, denoted $\mathsf{Ker}\, f$, is the tuple (K, k).

Definition 9. Let $\mathscr{C}$ be a category with terminal object 1 and $f : X \to Y$. The *cokernel* of f, denoted $\mathsf{Coker}\, f$, is the tuple (C, c) in the pushout

$$\begin{array}{ccc} X & \overset{f}{\longrightarrow} & Y \\ \downarrow & & \downarrow{\scriptstyle c} \\ 1 & \longrightarrow & C \end{array}$$

of f and the unique map $X \to 1$.

Sometimes we will write $\mathsf{Ker}\, f$ to denote just the object K, or just the map k, when the context is clear. We will abuse the definition of $\mathsf{Coker}\, f$ in the same way.

Kernels and cokernels may not exist. If they do, then they are unique up to unique isomorphism, since they are defined as limits [Le, Corollary 6.1.2].

The requirement that $\mathscr{C}$ has an initial object 0 is necessary to understand what is meant by 'the part of the domain that maps to zero'. This is the weakest assumption on $\mathscr{C}$ which allows us to interpret this statement.

Definition 10. Let $\mathscr{C}$ be an additive category and $f : X \to Y$ a morphism in $\mathscr{C}$. If it exists, the equaliser of f and 0 is the *kernel* of f:

$$K \xrightarrow{k} X \underset{0}{\overset{f}{\rightrightarrows}} Y.$$

Dually, the coequaliser of f and 0 is the *cokernel* of f:

$$X \underset{0}{\overset{f}{\rightrightarrows}} Y \xrightarrow{c} C$$

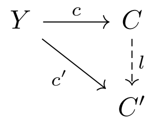
We can be even more explicit.

Definition 11. Let $\mathscr{C}$ be an additive category and $f : X \to Y$ a morphism in $\mathscr{C}$. The *kernel* of f is the tuple $(K, k' : K \to X)$ such that k is the morphism which is universal with respect to the property $fk = 0$, in the following sense: if $k' : K' \to X$ is a morphism such that $fk' = 0$, we have a unique map $l : K' \to K$ such that

$$\begin{array}{ccc} K' & & \\ {\scriptstyle l}\,\vdots\downarrow & \searrow^{k'} & \\ K & \underset{k}{\longrightarrow} & X \end{array}$$

commutes.

The *cokernel* of f is the tuple $(C, c : Y \to C)$ such that c is the morphism which is universal with respect to the property $cf = 0$. That is, if $c' : Y \to C'$ is another morphism such that $c'f = 0$, then we have a unique map $l : C \to C'$ such that



commutes.

Proposition 12. *The three definitions above of kernels and cokernels are equivalent, in an additive category.*

Proof. We need proposition 2 throughout. We will prove Definition 8 implies Definition 10. If we have a pullback

$$\begin{CD} K @>0>> 0 \\ @VkVV @VV0V \\ X @>>f> Y \end{CD}$$

then $K \xrightarrow{0} 0 \xrightarrow{0} Y = K \xrightarrow{0} Y = K \xrightarrow{k} X \xrightarrow{0} Y$. Thus,

$$K \xrightarrow{k} X \underset{0}{\overset{f}{\rightrightarrows}} Y$$

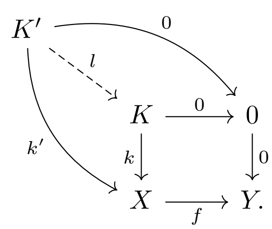
commutes. Now we show the universal property of the equaliser is satisfied. Suppose, we have another fork

$$K \xrightarrow{k'} X \underset{0}{\overset{f}{\rightrightarrows}} Y.$$

Then, by similar logic to above, we get a commutative square

$$\begin{CD} K' @>0>> 0 \\ @Vk'VV @VV0V \\ X @>>f> Y. \end{CD}$$

Using the universal property of pullbacks, there is a unique map l such that

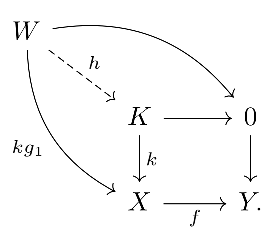


commutes. The bottom left triangle is exactly the universal property of the equaliser.

The rest of the proof is just more definition-unpacking and is therefore omitted. □

Proposition 13. *If $\mathscr{C}$ is a category with an initial object 0 and (K, k) is a kernel then k is monic. Dually, if $\mathscr{C}$ is a category with a terminal object 1 and (C, c) is a cokernel then c is epic.*

Proof. Suppose $g_1, g_2 : W \to K$ are maps such that $kg_1 = kg_2$. Then by the universal property of pushouts, there is a unique map h such that the diagram



commutes. But $h = g_1$ and $h = g_2$ both make the diagram commute. So $g_1 = g_2$. The second statement follows dually. □

The following propositions shows that these definitions generalise the traditional notion of kernels and cokernels.

Proposition 14. *The kernel of a morphism $f : G \to H$ in* ***Ab*** *is the subgroup $K \subset G$ which maps to $0 \in H$ together with the inclusion map i.*

Proof. Suppose we have a tuple $(K', k' : K' \to G)$ such that $fk' = 0$. This is exactly saying that $k'(K') \subset K$. So we have a unique map $k' : K' \to K$ as required. ☐

Proposition 15. *The cokernel of a morphism $f : G \to H$ in* ***Ab*** *is the quotient $C = H/\mathsf{Im}\, f$ together with the quotient map $c : H \to C$.*

Proof. Suppose we have a map $c' : H \to C'$ with $c'f = 0$. Define a map $l : C \to C'$ that sends an equivalence class $[h] \in C$ to $c'(h)$. Why is this well defined? If h and h' represent the same class $[h]$ then $h = h' + f(g)$ for some $g \in G$. So $c'(h) = c'(h') + c'f(g) = c'(h')$. Then the triangle

$$\begin{array}{ccc} H & \xrightarrow{c} & C \\ & \searrow^{c'} & \downarrow l \\ & & C' \end{array}$$

commutes. It is obvious that l is the unique map with this property. (Try defining another such map!) ☐

4 Abelian Categories

Definition 16. An additive category $\mathscr{C}$ is *abelian* if

1. every map has a kernel and a cokernel,
2. every monic is the kernel of some map, and
3. every epic is the cokernel of some map.

This is the standard definition of an abelian category. (For example, it is the definition given in [Fr].) It is hard to see how this generalises the category **Ab** of abelian groups.

There is another standard definition, which changes conditions 2. and 3.

Proposition 17. *An additive category is abelian if and only if*

1. *every map has a kernel and a cokernel,*

2^*. *every monic is the kernel of its cokernel, and*

3^*. *every epic is the cokernel of its kernel.*

The final definition of an abelian category, given below, seems to be a more intuitive generalisation of **Ab** than the previous two definitions.

Proposition 18. *An additive category $\mathscr{C}$ is abelian if and only if for every morphism $f : X \to Y$, there exists a sequence*

$$K \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} C \tag{1}$$

with the following properties:

1. *$ji = f$,*
2. *$(K, k) = \mathsf{Ker}\, f, (C, c) = \mathsf{Coker}\, f$,*
3. *$(I, i) = \mathsf{Coker}\, k$ and $(I, j) = \mathsf{Ker}\, c$.*

This is Etingof definition of abelian categories, given in [Et]. The sequence 1 is unique up to unique isomorphism, since kernels and cokernels are unique up to unique isomorphism. We call the sequence 1 the canonical decomposition (or image factorisation) of f. The object I is called the image of f and is denoted by $\mathsf{Im}\, f$.

The proof of these propositions is in appendix A. See Proposition 34.

As an example, the category R-**Mod** of all left (or right) modules over a ring R is an abelian category. We can generalise this: if $\mathscr{C}$ is a small **Ab**-category and $\mathscr{D}$ is an abelian category, then the additive functors from $\mathscr{C}$ to $\mathscr{D}$ form an abelian category. We get the previous example by considering a ring R as an **Ab**-category and an R-module as an additive functor from R to **Ab**.

5 Mitchell's Embedding Theorem

Theorem 19 [Mi]**.** *Let $\mathscr{C}$ be an abelian category and let $\mathscr{C}_0 \subset \mathscr{C}$ be a small abelian subcategory. There is a ring R and an exact, fully faithful functor from $\mathscr{C}$ into R-**Mod** which embeds $\mathscr{C}_0$ as a full subcategory.*

This formulation of the theorem is given in [Al]. We define what an exact functor is in section 7. For now, it is sufficient to note that this theorem gives a particularly nice equivalence between a small abelian category and a full subcategory of R-**Mod** for some ring R. As was mentioned in the introduction, this allows a more hands-on study of abelian categories. The abstract definitions of kernel, cokernel, morphisms, monics, epics, quotients, etc. simplify to their well known and well understood traditional definitions. For example, in section 9, we use Mitchell's embedding theorem to generalise the standard proof of the snake lemma to abelian categories. Other results in homological algebra, for example the five lemma, can also be generalised in this way.

However, there are important limitations to this result. The obvious one is that it only holds for small abelian categories. The non-standard statement of the theorem given above goes some way to remedying this: If we only need to consider a small part of the category at a time (for example, if we're proving a result about an (arbitrary) but finite number of objects and morphisms, as is usually the case), then we can still embed this into a category of modules to get the result.

Another limitation is that, without looking into the proof, we have no idea what the ring R is. It turns out that the ring R is not unique and there are often no manageable choices of R [Et, p.3].

Moreover, in some sense Mitchell's embedding theorem goes against the 'category theoretic perspective' (c.f. the nPOV on nLab [nP]). Category theory provides a different foundation of mathematics, which ignores the notions of sets and elements. Many would say that this has proven to be very useful in revealing the nature of mathematical objects. But Mitchell's embedding theorem goes the other way — from category theoretic perspective to the world of sets and elements.

Finally, it turns out that all the diagram lemmas in homological algebra can be proved directly for abelian categories by working with generalised elements (c.f. [nE]), without needing to embed into R-**Mod**. In fact, there appears to be no well known results about abelian categories whose proof relies on Mitchell's theorem. See Brandenburg's great answer [Br] on Maths Stackexchange for a more detailed discussion. It is (presumably) in this sense that Etingof says that the embedding theorem is only 'psychologically useful' [Et, p.3].

6 Monics and Epics in Additive Categories

These next two sections will give some properties of additive and abelian categories while providing some of the background necessary to prove Mitchell's embedding theorem.

Proposition 20. *In an **Ab**-category, $f : X \to Y$ is monic if and only if $fg = 0$ implies $g = 0$ for every morphism $g : W \to X$. Dually, f is epic if and only if $hf = 0$ implies $h = 0$ for all maps $h : Y \to Z$.*

Proof. Suppose f is monic. By definition, this means that if $fg_1 = fg_2$ then $g_1 = g_2$ for $g_1, g_2 : W \to X$. If $fg = 0$ then $fg = f0$ by Proposition 2 and consequently $g = 0$.

Suppose $fg = 0$ implies $g = 0$ for every morphism $g : W \to X$. Further suppose $fg_1 = fg_2$. Then $f(g_1 - g_2) = 0$ and so $g_1 - g_2 = 0$.

The second statement can be proven analogously. ☐

Corollary 21. *In an additive category, $f : X \to Y$ is monic if and only if it has kernel $(0, 0 \xrightarrow{0} X)$ and $g : X \to Y$ is epic if and only if it has a cokernel $(0, Y \xrightarrow{0} 0)$.*

Proof. Suppose f is monic. We want to show that for every morphism $g : W \to X$ with $fg = 0$, there is a unique map $h : W \to 0$ making the diagram

$$\begin{array}{ccc} W & & \\ {\scriptstyle h}\,\downarrow & \searrow^{g} & \\ 0 & \xrightarrow[0]{} & X \end{array}$$

commute. We have $g = 0$ by Proposition 20. Thus, $0 : W \to 0$ is a morphism satisfying the commutative diagram above. Since 0 is a zero object, 0 is the unique such map.

Now suppose f has kernel $(0, 0 \xrightarrow{0} X)$. We want to show that for every morphism $g : W \to X$ with $fg = 0$, we have $g = 0$. The commutative relation of kernel $(0, 0 \xrightarrow{0} X)$ states that there exists a morphism $h : W \to 0$ such that $g = 0h$. But by Proposition 2, this implies $g = 0$. Again, the proof of the second statement is dual to the proof of the first. ☐

Theorem 22. *In an abelian category $\mathscr{C}$, a map is both monic and epic if and only if it is an isomorphism.*

The if direction is straightforward and holds in any category. The only if direction is theorem 2.12 of [Fr].

Proof. Note that, in an arbitrary category, if the equaliser $f : X \to Y$ of a pair of maps $g_1, g_2 : Y \to Z$ is epic, then it is an isomorphism. Why? If $g_1 f = g_2 f$, then f epic implies $g_1 = g_2$. Thus, id_Y is an equaliser of g_1, g_2. But equalisers are unique up to a unique isomorphism that commutes with f and id_Y. Thus, f is an isomorphism. Now, every monic is a kernel, so it is an equaliser. □

There are a lot more important results about epics and monics, in particular how they interact with kernels, cokernels and images. However, they are analogous to well known results about abelian groups, so we have not included them. See chapter 2 of [Fr].

Definition 23. Let X be an object of a category $\mathscr{C}$. A *subobject* of X is an object Y and a monomorphism $i : Y \to X$. A *quotient* object of X is an object Z with an epimorphism $p : X \to Z$.

One great thing about abelian categories is that we have a good notion of quotients X/Y of objects. If Y is a subobject of X define the quotient object X/Y to be the cokernel of the monomorphism $f : Y \to X$.

7 Exact Sequences

Definition 24. A sequence of objects and morphisms

$$... \to X_n \xrightarrow{f_n} X_{n+1} \xrightarrow{f_{n+1}} X_{n+2} \to ...$$

is *exact at degree n* if $\mathsf{Im}\, f_{n-1} \cong \mathsf{Ker}\, f_n$. A sequence is *exact* if it is exact at every degree.

An exact sequence is *short* if it is of the form

$$... \to 0 \to ... \to 0 \to X \to Y \to Z \to 0 \to ... \to 0 \to ...$$

Generally, we ignore the 0 objects on the left and right of the sequence and write a short exact sequence as $0 \to X \to Y \to Z \to 0$.

Definition 25. An additive functor $F : \mathscr{C} \to \mathscr{D}$ between abelian categories is *left exact* if for all short exact sequences $0 \to X \to Y \to Z \to 0$ in $\mathscr{C}$, the sequence $0 \to F(X) \to F(Y) \to F(Z)$ is exact. Similarly, F is *right exact* if the sequence $F(X) \to F(Y) \to F(Z) \to 0$ is exact. We say F is *exact* if it is both left and right exact.

It turns out the if a functor preserves short exact sequences, then it preserves arbitrary exact sequences:

Theorem 26. *If $F : \mathscr{C} \to \mathscr{D}$ is an exact functor between abelian categories and*

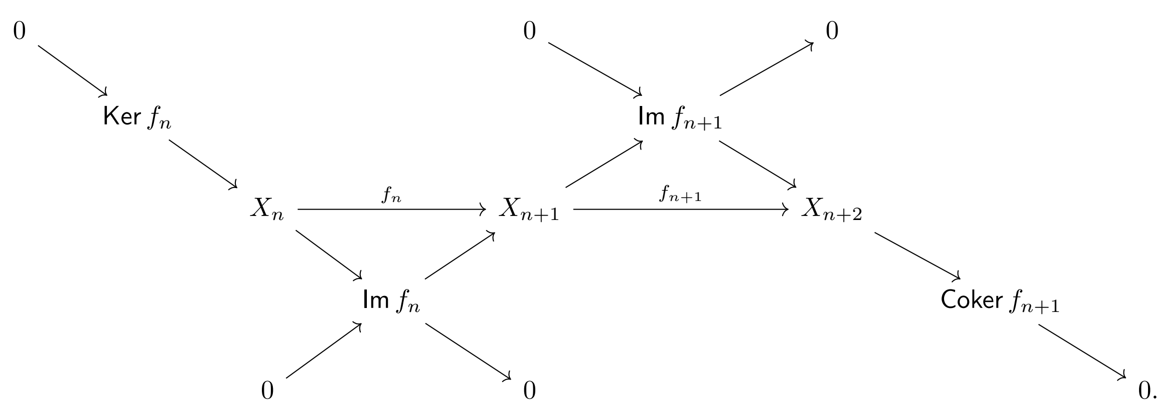
$$... \to X_n \xrightarrow{f_n} X_{n+1} \xrightarrow{f_{n+1}} X_{n+2} \to ...$$

is an exact sequence in $\mathscr{C}$, then its image

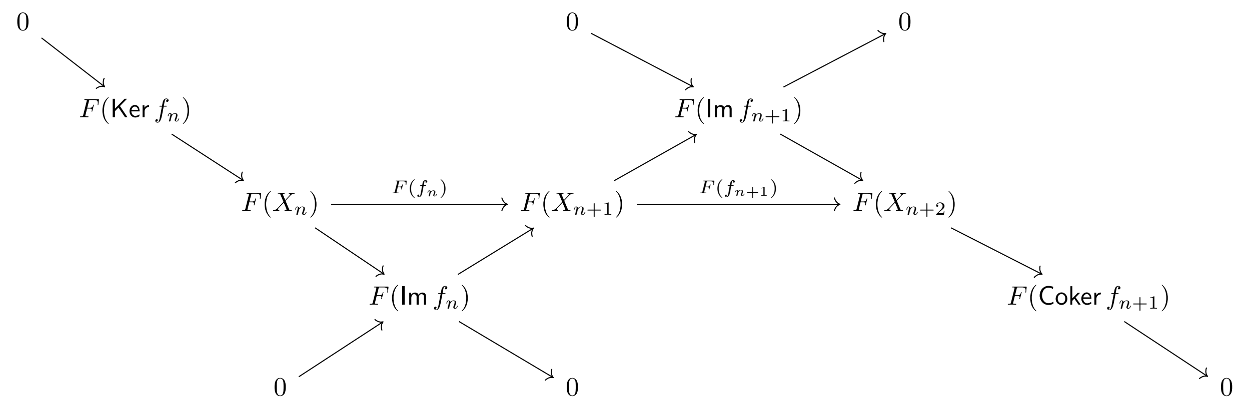
$$... \to F(X_n) \xrightarrow{F(f_n)} F(X_{n+1}) \xrightarrow{F(f_{n+1})} F(X_{n+2}) \to ...$$

is exact.

Proof sketch (taken from [Fi]). We want to prove exactness at X_{n+1}. We can construct a commutative diagram



The diagonals are exact. (Proving this is a straightforward exercise.) The image of this diagram in $\mathscr{D}$



also has exact diagonals. (Here we used the fact that $F(0) \cong 0$, proved in Proposition 7.)

We can compute

$$\operatorname{Im} F(f_n) = \operatorname{Im}\, (F(X_n) \to F(\operatorname{Im} f_n) \to F(X_{n+1})) = \operatorname{Im}\, (F(\operatorname{Im} f_n) \to F(X_{n+1}))\,,$$

since $F(X_n) \to F(\operatorname{Im} f_n)$ is epic. Also

$$\operatorname{Im}\, (F(\operatorname{Im} f_n) \to F(X_{n+1})) = \operatorname{Ker}\, (F(X_{n+1}) \to F(\operatorname{Im} f_{n+1})) = \operatorname{Ker}\, (F(X_{n+1}) \to F(\operatorname{Im} f_{n+1}) \to F(X_{n+2}))\,.$$

since $F(\operatorname{Im} g) \to F(X_{n+2})$ is monic. Thus, $\operatorname{Im} F(f_n) = \operatorname{Ker} F(f_{n+1})$. □

Exact functors are the natural choice of 'nice' maps between abelian categories, since they preserve all of the structure of abelian categories:

Theorem 27. *If $F : \mathscr{C} \to \mathscr{D}$ is an exact functor between abelian categories then*

1. *F preserves all finite coproducts,*
2. *F preserves all finite products,*
3. *F preserves the zero object and binary direct sums,*
4. *F preserves all finite limits,*
5. *F preserves all finite colimits,*
6. *F preserves all kernels,*
7. *F preserves all cokernels, and*
8. *F preserves chain homology.*

For a functor between additive categories, properties 1. to 3. are equivalent to being additive. Property 4. (resp. 5.) is equivalent to left (resp. right) exactness, for functors between abelian categories. Additive functors between abelian categories satisfying properties 6. (resp. 7.) are precisely the left (resp. right) exact functors.

8 Proof of Mitchell's embedding theorem

This section is based on [Al]. Throughout this section, let $\mathscr{C}$ be a locally small abelian category unless stated otherwise.

Lemma 28 (Theorem 1.6.11 of [We])**.** *The sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact if the induced sequence*

$$\mathscr{C}(C \to X) \xrightarrow{f\circ -} \mathscr{C}(C \to Y) \xrightarrow{g\circ -} \mathscr{C}(C \to Z) \tag{2}$$

is exact for every $C \in \mathscr{C}$.

Proof (sketch). First, setting $C = X$ gives $gf = gf\mathsf{id}_X = 0$ by exactness of equation 2. So $\mathsf{Im}\, f$ is a subobject of $\mathsf{Ker}\, g$ (easy exercise: the unique map $h : \mathsf{Im}\, f \to \mathsf{Ker}\, g$ given by the universal property of $\mathsf{Ker}\, g$ is monic since $\mathsf{Im}\, f \to Y$ is monic).

Now set $C = \mathsf{Ker}\, g$. Since $i : \mathsf{Ker}\, g \to Y$ satisfies $gi = 0$, there is some $h : \mathsf{Ker}\, g \to X$ such that $fh = i$. We know $\mathsf{Ker}\, g$ is a subobject of $\mathsf{Im}\, i$, which is a subobject of $\mathsf{Im}\, f$ (another easy exercise). Thus, $\mathsf{Ker}\, g$ is a subobject of $\mathsf{Im}\, f$. The result then follows. □

Lemma 29 (Theorem 7.11 of [Fr]). *If a functor $F : \mathscr{C} \to \mathbf{Ab}$ is faithful and preserves monics, then it is exact.*

Proof (sketch). Take a short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathscr{C}$. The representation functor $H^- : \mathscr{C}^{\mathrm{op}} \to [\mathscr{C}, \mathbf{Ab}]$ (c.f. [Le, chapter 4]) is left exact. Thus, $0 \to H^Z \to H^Y \to H^X$ is exact in the functor category $[\mathscr{C}, \mathbf{Ab}]$.

Now, the functor category $[\mathscr{C}, \mathbf{Ab}]$ is abelian. That is, the natural transformations $G \Rightarrow H$ for two functors $G, H : \mathscr{C} \to \mathbf{Ab}$ form an abelian group. Thus, we get a (contravariant) functor $[\mathscr{C}, \mathbf{Ab}](- \to H)$ from $[\mathscr{C}, \mathbf{Ab}]$ to $\mathbf{Ab}$. This is exact (see [Fr, section 5.1]).

Therefore, the sequence

$$[\mathscr{C}, \mathbf{Ab}](H^X \to F) \longrightarrow [\mathscr{C}, \mathbf{Ab}](H^Y \to F) \longrightarrow [\mathscr{C}, \mathbf{Ab}](H^Z \to F) \longrightarrow 0$$

is exact. By a modified version of Yoneda lemma (Theorem 5.32 of [Fr]) for abelian categories, this is isomorphic to the sequence $F(X) \to F(Y) \to F(Z) \to 0$. Thus, F is right exact.

Finally, a right-exact functor F is exact if it preserves monics. Why? For any short exact sequence, $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$, f is monic. If $F(f)$ is also monic, then its kernel is $(0, 0 \to X)$ (Corollary 21). Thus, we get that $0 \to F(X) \xrightarrow{F(f)} Y \xrightarrow{F(g)} Z \to 0$ is exact. (It is not hard to see that the converse is also true: if a right-exact functor is exact then it preserves monics.) □

Let $[\mathscr{C}, \mathbf{Ab}]_{\mathrm{add}}$ be the category of additive functors from $\mathscr{C}$ to $\mathbf{Ab}$.

Recall that a category is complete if it has all equalisers and all (small) products.

Theorem 30 (Theorems 7.31, 7.32, 7.33 of [Fr]). *Let $\mathscr{L} \subset [\mathscr{C}, \mathbf{Ab}]_{\mathrm{add}}$ be the full subcategory of left-exact functors. Then $\mathscr{L}$ is complete and abelian. The image of the representation functor $H : \mathscr{C}^{\mathrm{op}} \to [\mathscr{C}, \mathbf{Ab}]$ is in $\mathscr{L}$ and is exact and fully faithful. Moreover, there is an object $F \in \mathscr{L}$ such that*

$$\mathscr{L}(- \to F) : \mathscr{L} \longrightarrow \mathbf{Ab}$$

is faithful and exact.

We call F an injective cogenerator.

Proof (sketch). That $\mathscr{L}$ is abelian is verified in [Fr, Theorem 7.31]. Next, we need to show that $\mathscr{L}$ is complete. First we show that $\mathscr{L}$ has all equalisers. In fact, any additive category with kernels has all equalisers: For $f, g : X \to Y$, their equaliser is kernel of their difference $f - g$. Why? The equaliser of f, g, if it exists, is the map h with $hf = hg$ and is universal with respect to this property. But this is exactly the same as asking for $h(f - g) = 0$, and $\mathsf{Ker}\,(f - g)$ is exactly the map with this universal property. Thus, we need only check that $\mathscr{L}$ has all products. This is straightforward consequence of the additive structure of $[\mathscr{C}, \mathbf{Ab}]_{\mathrm{add}}$.

To verify that the image of H is in $\mathscr{C}$, we need to check that H^X is an additive, left exact functor. In fact, any left exact functor is additive [Fr, Theorem 3.12] and [Fr, example on p.65] shows that H^X is left exact.

By definition H is fully faithful. Now we will show that H is exact. Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence in $\mathscr{C}$. We need to verify that $0 \to H^Z \to H^Y \to H^X \to 0$ is exact. This is equivalent to checking that

$$0 \longrightarrow [\mathscr{C}, \mathbf{Ab}](H^X \to F) \longrightarrow [\mathscr{C}, \mathbf{Ab}](H^Y \to F) \longrightarrow [\mathscr{C}, \mathbf{Ab}](H^Z \to F) \longrightarrow 0$$

is exact for F an injective cogenerator in $\mathscr{L}$. By the Yoneda lemma (Theorem 5.34 of [Fr]), this is isomorphic to $0 \to F(X) \to F(Y) \to F(Z) \to 0$, which is exact if F is exact. Using Lemma 29, we can prove that F is exact. Thus, H is exact, as required.

Finally, we need to check that $\mathscr{L}$ actualy has an injective cogenerator F. (Note that we relied on the existence of F in the paragraph above.) This is the second half of Theorem 7.32 in [Fr]. □

Lemma 31 (Theorem 4.44 of [Fr]). *Let $\mathscr{D}$ be a complete abelian category with an object P such that $\mathscr{D}(P \to -) : \mathscr{D} \to \mathbf{Ab}$ is faithful and exact. (Such an object P is called a projective generator.) For every small abelian subcategory $\mathscr{D}_0 \subset \mathscr{D}$, there is a ring R and an exact, fully faithful functor from $\mathscr{D}_0$ into R-$\mathbf{Mod}$.*

We will not prove this difficult lemma.

Proof of Mitchell's embedding theorem. Let $\mathscr{C}$ be an abelian category. Without loss of generality, we can assume $\mathscr{C}$ is locally small. Let $\mathscr{C}_0 \subset \mathscr{C}$ be a small abelian subcategory.

By Theorem 30 we have an exact, fully faithful, covariant functor $H : \mathscr{C} \to \mathscr{L}^{\text{opp}}$. Additionally, $\mathscr{L}$ is a complete abelian category with an injective cogenerator.

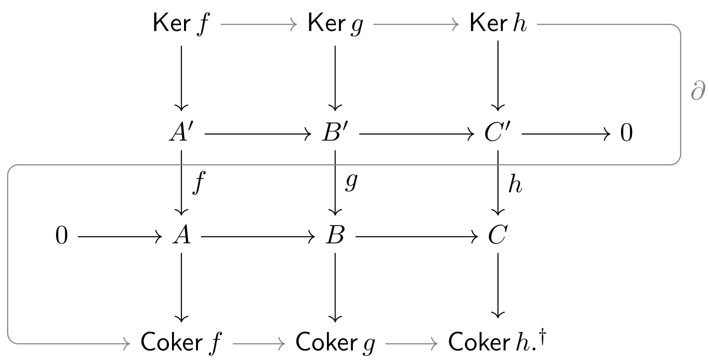
By the same reasoning as in the proof of Theorem 30, all abelian categories have all coequalisers. Moreover, products and coproducts are the same in abelian categories. Thus, an abelian category is complete if and only if it is cocomplete.

So $\mathscr{L}^{\text{opp}}$ is a complete abelian category with a projective generator. Applying Lemma 31 tells us that the small abelian subcategory $H(\mathscr{C}_0)$ of $\mathscr{L}^{\text{opp}}$ embeds into R-**Mod**, giving us the result. □

9 The Snake Lemma

We conclude by proving the snake lemma for abelian categories. Since we now have Mitchell's embedding theorem, it suffices to prove the lemma for the category R-**Mod**, as we will explain in the proof below.

Theorem 32 (Snake Lemma)**.** *Let $\mathscr{C}$ be an abelian category. Suppose the diagram in black below commutes, with exact rows. Then the sequence in gray exists and is exact.*



Proof (sketch). Take a small subcategory $\mathscr{C}_0$ containing the commutative diagram. Embed this into R-**Mod** using Mitchell's embedding theorem. Since the embedding is exact, the image of the diagram will be commutative, with exact rows. Moreover, if we can establish

$$\mathsf{Ker}\, a \to \mathsf{Ker}\, b \to \mathsf{Ker}\, c \xrightarrow{\delta} \mathsf{Coker}\, a \to \mathsf{Coker}\, b \to \mathsf{Coker}\, c$$

in R-**Mod**, then it must also hold in $\mathscr{C}$.

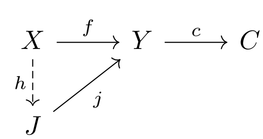
Thus, we need only prove the snake lemma for the category R-**Mod**. This is a standard proof — any book in homological algebra will contain it. See for example [La, chapter 3, section 9]. □

10 Appendix A: Equivalent Definitions of an Abelian Category

Lemma 33. *Suppose we have a morphism $f : X \to Y$ in an additive category. Further, suppose* $\mathsf{Coker\,Ker}\, f = (I, i)$ *and* $\mathsf{Ker\,Coker}\, f = (J, j)$ *exist. Then there exists a map $\overline{f}$ such that f decomposes as*

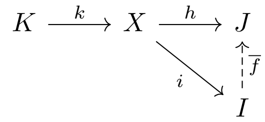
$$X \xrightarrow{i} I \xrightarrow{\overline{f}} J \xrightarrow{j} Y.$$

Proof. Let $\mathsf{Coker}\, f = (C, c)$ and $\mathsf{Ker}\, f = (K, k)$. We have that $cf = 0$, so by the universal property of $\mathsf{Ker\,Coker}\, f$, there is a map $h : X \to J$ such that



†Code by TeX Stackexchange user Loop Space [Lo].

commutes. Now, if we can show that $hk = 0$, then the universal property of $\mathsf{Coker}\,\mathsf{Ker}\, f$ implies that there is a map $\overline{f} : I \to J$ such that



commutes. Combining these two diagrams, we get $j\overline{f}i = jh = f$ as required.

It remains to show that $hk = 0$. We know $jhk = fk = 0$. But j is a kernel, so it is monic by Proposition 13. Thus, $jkh = 0 = j0$ implies $kh = 0$ by Proposition 20. ☐

Proposition 34. *For an additive category $\mathscr{C}$, the following statements are equivalent*

(a) $\mathscr{C}$ is abelian.

(b) $\mathscr{C}$ has the following properties:

1. *every map has a kernel and a cokernel,*

2^*. *every monic is the kernel of its cokernel, and*

3^*. *every epic is the cokernel of its kernel.*

(c) For every morphism f, the induced map $\overline{f}$: $\mathsf{Coker}\,\mathsf{Ker}\, f \to \mathsf{Ker}\,\mathsf{Coker}\, f$ exists and is an isomorphism.

(d) For every morphism $f : X \to Y$, there exists a sequence

$$K \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} C$$

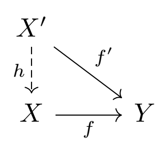
with the following properties:

1. *$ji = f$,*
2. *$(K, k) = \mathsf{Ker}\, f, (C, c) = \mathsf{Coker}\, f$,*
3. *$(I, i) = \mathsf{Coker}\, k$ and $(I, j) = \mathsf{Ker}\, c$.*

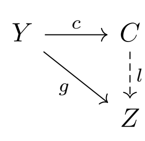
Proof. "$(a) \Rightarrow (b)$": We will prove that

1. if f has cokernel (C, c) and is the kernel of some map, then f is the kernel of (C, c), and
2. if f has kernel (K, k) and is the cokernel of some map, then f is the cokernel of (K, k).

First, we will show 1. Let $f : X \to Y$ have cokernel (C, c) and suppose it is the kernel of some map $g : Y \to Z$. We need to prove that, for every map $f' : X' \to Y$ with $cf' = 0$, there exists a unique map $h : X' \to X$ such that



commutes. Since (C, c) is the cokernel of f, there exists a unique map $l : C \to Z$ such that

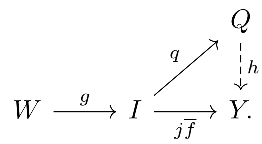


commutes. Thus, $gf' = lcf' = l0 = 0$. Since f is the kernel of g, it follows that h exists and is unique. The proof of 2. is dual to the proof of 1.

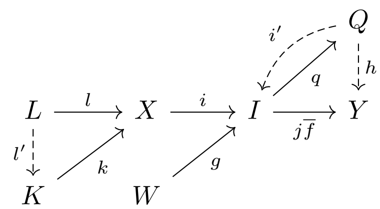
"$(b) \Rightarrow (a)$": This is trivial.

"$(a) \Rightarrow (c)$": Using Lemma 33, we decompose f as $X \xrightarrow{i} I \xrightarrow{\overline{f}} J \xrightarrow{j} Y$. Furthermore, we write (K, k) for $\mathsf{Ker}\, f$. We will show that $j\overline{f}$ is monic and $\overline{f}i$ is epic. This imples $\overline{f}$ is both monic and epic, and therefore an isomorphism by Theorem

22. Consider some map $g : W \to I$ such that $j\overline{f}g = 0$. We want to show $g = 0$. Let (Q, q) be the cokernel of g and let $h : Q \to Y$ be the unique map such that $j\overline{f} = hq$:



By Proposition 13, qi is epic. Thus, there exists some $l : Z \to X$ such that (Q, qi) is the cokernel of l. Then $fl = j\overline{f}il = hqil = h0 = 0$. This allows us to apply the universal property of $\mathsf{Ker}\, f$ and find a map $l' : L \to K$ such that $l = kl'$. Then $il = ikl' = 0l' = 0$ since i is the cokernel of k. Apply the universal property of $\mathsf{Coker}\, l = (Q, qi)$ to get a map $i' : Q \to X$ such that $i = i'qi$.



So $i = (i'q)i$. Since i is epic (Proposition 13), $i'q = \mathsf{id}_I$. It follows q is monic. Finally $qg = 0$ implies $g = 0$, as required. The dual of this argument shows that $\overline{f}i$ is epic.

"$(c) \Rightarrow (d)$": This is trivial.

"$(d) \Rightarrow (a)$": Condition 1 in the definition of an abelian category (Definition 16) is easily satisfied. Suppose $f : X \to Y$ is a monic. We want to show that $\mathsf{Ker}\,\mathsf{Coker}\, f = (X, f)$. By Corollary 21, it has kernel $(0, 0 \to X)$. It is easy to see that the cokernel of $0 \to X$ is (X, id_X). Thus, the canonical decomposition of f (equation 1) looks like

$$0 \to X \xrightarrow{\mathsf{id}_X} X \xrightarrow{j} Y \to C.$$

By property 1. of (d), j must be equal to f. Then by property 3. we get that $(X, f) = \mathsf{Ker}\,\mathsf{Coker}\, f$, as required. □

By Lemma 33, we can see that it makes sense to define the image of a map f as the object J in the decomposition $X \xrightarrow{i} I \xrightarrow{\overline{f}} J \xrightarrow{j} Y$. That, is $\mathsf{Im}\, f = \mathsf{Ker}\,\mathsf{Coker}\, f$. Dually, we can define the coimage as the object I. This gives us a final characterisation of abelian categories: an additive category is abelian if and only if all coimages are isomorphic to their corresponding image via $\overline{f}$.

References

[Al] G. Aly, *Abelian Categories and the Freyd-Mitchell Embedding Theorem*, 2009 (available online at `http://u.arizona.edu/~geillan/research/ab_categories.pdf`).

[Br] M. Brandenburg, *Answer to 'Applications of Mitchell's embedding theorem'*, Maths Stackexchange, 2014 (available online at `https://math.stackexchange.com/a/362159`).

[Bu] D. A. Buchsbaum, *Exact categories and duality*, Transactions of the American Mathematical Society **80** (1955), 1–34.

[Et] P. Etingof, S. Gelaki, D. Nikshych and V. Ostrik, *Tensor Categories*, American Mathematical Society, Providence, 2015 (available online at `http://www-math.mit.edu/~etingof/egnobookfinal.pdf`).

[Fi] B. Fitzpatrick, *Answer to 'Showing exact functors preserve exact sequences (abelian categories, additive functors, and kernels)'*, Maths Stackexchange, 2014 (available online at `https://math.stackexchange.com/a/803829`).

[Fr] P. Freyd, *Abelian Categories*, Harper & Row, New York, 1964.

[Gr] A. Grothendieck, *Sur quelques points d'algèbre homologique*, The Tohoku Mathematical Journal, second series **9** (1957), 119–221.

[La] S. Lang, *Algebra*, Springer, New York, 2002.

[Le] T. Leinster, *Basic Category Theory*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2014 (available online at `https://arxiv.org/abs/1612.09375`).

[Lo] TeX Stackexchange user Loop Space, *Answer to 'How do you draw the "snake" arrow for the connecting homomorphism in the snake lemma?'*, TeX Stackexchange, 2010 (updated 2017) (available online at `https://tex.stackexchange.com/a/3894`).

[nK] nLab, *Kernel*, 2016 (available online at `https://ncatlab.org/nlab/show/kernel`).

[nE] nLab, *Element in an Abelian Category*, 2015 (available online at `https://ncatlab.org/nlab/show/element+in+an+abelian+category`).

[nP] nLab, *nPOV*, 2014 (available online at `https://ncatlab.org/nlab/show/nPOV`).

[nC] nLab, *Cokernel*, 2013 (available online at `https://ncatlab.org/nlab/show/cokernel`).

[Mi] B. Mitchell, *The full imbedding theorem*, American Journal of Mathematics **86:3** (1964), 619-637.

[We] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge University Press, Cambridge, 1994.