Hausdorff and Similarity Dimensions

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1 Introduction

The concept of dimension is fundamental to mathematics. Intuitively, if you scale an object O by some factor c, then the number of times N the original object O fits inside the scaled object O' is propotional to c^D, where D is the dimension of O. (Think of N as the relative size of O'.) Scaling a line segment l by a factor of 2 gives two copies of l. Thus, l has dimension 1. On the other hand, scaling a square or a cube by a factor of 2 gives 4 and 8 copies, respectively. So squares have dimension 2 and cubes dimension 3.

There are a number of formal definitions of dimension, for example the Lebesgue covering dimension or the inductive dimension. For a reference on these and other topological dimensions see [8]. However, the intuitive dimension of a fractal or other structurally complicated sets is often non-integral and does not coincide with these formal definitions. To counter this, we will introduce the Hausdorff and similarity dimensions, which, unlike the above notions of dimension, can take non-integral values. We will develop the theory behind these dimensions, with particular regard to fractals, and conclude by showing that these definitions coincide under mild conditions.

2 The Hausdorff Dimension

Definiton 1. *Let $k \geq 0$ be a non-negative real number. For every $\delta > 0$ and every subset $E \subset \mathbb{R}^d$, define*

$$\mathcal{H}_\delta^k(E) = \inf \left\{ \sum_{i=1}^{\infty} \alpha_k 2^{-k} (\operatorname{diam} E_i)^k : E \subset \bigcup_{i=1}^{k} E_i, \operatorname{diam} E_i \leq \delta \right\},$$

where $\operatorname{diam} S = \sup\{|x-y| : x, y \in S\}$ denotes the diameter of the set S and α_k is a suitable normalising constant, defined 2 paragraphs below. Define the ***exterior k-dimensional Hausdorff measure*** *as the limit of $\mathcal{H}_\delta^k$ as δ tends to 0:*

$$\mathcal{H}^k(E) = \lim_{\delta \to 0} \mathcal{H}_\delta^k(E).$$

In other words, $\mathcal{H}^k(E)$ is defined in the following way: consider covers of E by countable families $\{E_j\}$ of sets with diameter less than δ and take the infimum of the sum $\sum_j \alpha_k 2^{-k} (\operatorname{diam} E_j)^k$. Then $\mathcal{H}^k(E)$ is the limit of these infimums as δ tends to 0.

The normalising constant is $\alpha_k = \Gamma(\frac{1}{2})^k / \Gamma(\frac{k}{2}+1)$, where Γ is Euler's gamma function. In particular, for integral k, we have $\alpha_k = \lambda^k(\{x \in \mathbb{R}^k : |x| \leq 1\})$, where λ^k is the k-dimensional Lebesgue measure. That is, α_k is equal to the volume of the k-dimensional unit ball.

Although not obvious, it follows that for any Borel set $E \subset \mathbb{R}^d$, we have $\lambda^d(E) = \mathcal{H}^d(E)$. Moreover, the fact that $\mathcal{H}_k(E)$ is indeed an exterior measure is not immediate. Proofs of these results are given in [9, section 7.1] along with the result that $\mathcal{H}_k(E)$ is countably additive for Borel sets and thus, a measure when restricted to Borel sets.

We will define the Hausdorff dimension of $E \subset \mathbb{R}^d$ as the unique real $k \geq 0$ with $\mathcal{H}^j(E) = \infty$ for $j < k$ and $\mathcal{H}^j(E) = 0$ for $j > k$. To make this definition, we need the following lemma:

Lemma 1. *If $\mathcal{H}^k(E) < \infty$ and $j > k$, then $\mathcal{H}^j(E) = 0$. Also, if $\mathcal{H}^k(E) > 0$ and $j < k$, then $\mathcal{H}^j(E) = \infty$.*

Proof. For the first result, suppose $\mathcal{H}^k(E) < \infty$ and $j > k$. Fix $\delta > 0$. If $\operatorname{diam} E_i \leq \delta$ then,

$$(\operatorname{diam} E_i)^j = (\operatorname{diam} E_i)^{j-k}(\operatorname{diam} E_i)^k \leq \delta^{j-k}(\operatorname{diam} E_i)^k.$$

Note $\mathcal{H}^k_\delta(E) \leq \mathcal{H}^k(E)$. Hence,

$$\mathcal{H}^j_\delta(E) \leq \delta^{j-k}\frac{\alpha_j 2^{-j}}{\alpha_k 2^{-k}}\mathcal{H}^k_\delta(E) \leq \delta^{j-k}\frac{\alpha_j 2^{-j}}{\alpha_k 2^{-k}}\mathcal{H}^k(E).$$

Taking $\delta \to 0$, we get that $\mathcal{H}^j(E) = 0$. The second result follows similarly: Suppose $j < k$ and $\mathcal{H}^k(E) > 0$. Fix $\delta > 0$. We get

$$(\operatorname{diam} E_i)^k \leq \delta^{k-j}(\operatorname{diam} E_i)^j,$$

by the same reasoning as above. Then

$$\mathcal{H}^j(E) \geq \mathcal{H}^j_\delta(E) \geq \delta^{j-k}\frac{\alpha_j 2^{-j}}{\alpha_k 2^{-k}}\mathcal{H}^k_\delta(E).$$

As $\delta \to 0$, we have $\delta^{j-k} \to \infty$. Hence, $\mathcal{H}^j(E) = \infty$. □

Definiton 2. *The **Hausdorff dimension** of $E \subset \mathbb{R}^d$, written $\dim E$, is the unique k given by*

$$k = \sup\{j : \mathcal{H}^j(E) = \infty\} = \inf\{j : \mathcal{H}^j(E) = 0\}.$$

Note that $H^k(E)$ can take any value in $[0, \infty]$ for E with $\dim E = k$.

The Hausdorff dimension is central to the theory of fractals. In fact, Mandelbrot defines fractals as objects with non-integral Hausdorff dimension. See [6] and [9, chapter 7] for treatments on fractals using Hausdorff dimension.

3 Similitudes and Invariant Sets

Before we can define the similarity dimension of an object, we need to lay some general groundwork.

Definiton 3. *Let (X, d) be a metric space. The **Lipschitz constant** of $F : X \to X$ is*

$$\operatorname{Lip} F = \sup_{x \neq y}\frac{d(F(x), F(y))}{d(x, y)}.$$

*We say F is **Lipschitz** if $\operatorname{Lip} F < \infty$ and F is a **contraction** if $\operatorname{Lip} F < 1$.*

Definiton 4. *A function $S : \mathbb{R}^d \to \mathbb{R}^d$ is a **similitude** if there is some fixed $r \in \mathbb{R}$ such that $|S(x) - S(y)| = r|x - y|$ for all $x, y \in \mathbb{R}^d$.*

Note that a similitude S is Lipschitz with $\operatorname{Lip} S$ equal to the constant r in $|S(x) - S(y)| = r|x - y|$. From herein, $\mathcal{S} = \{S_1, ..., S_N\}$ is a finite set of similitudes where $\operatorname{Lip} S_i = r_i < 1$. (So S_i is also a contraction.) For arbitrary $E \subset \mathbb{R}^d$, define $\mathcal{S}(E) = \bigcup_{i=1}^N S_i(E)$.

Theorem 1. *There exists a unique compact set $K \subset \mathbb{R}^d$ which is invariant with respect to $\mathcal{S}$, in the sense that $\mathcal{S}(K) = K$.*

Like many of the results and definitions in this essay, this theorem can be generalised to an arbitrary complete metric space. See [5] for the more general treatment.

We denote the unique such K by $|\mathcal{S}|$. A typical way of constructing a fractal X is to specify a set of similitudes $\mathcal{S}$ and then define X to be $|\mathcal{S}|$. The similitudes describe how the fractal $|\mathcal{S}|$ is 'self-similar' (see [6]).

Since we are now in a position to define the similarity dimension, we will delay the proof of this theorem until section 5.

4 The Similarity Dimension

Consider the function $f(t) = \sum_{i=1}^N r_i^t$. It's easy to see that $f(0) = N$ and f is strictly decreasing continuous function approaching 0 as $t \to \infty$. Thus, there exists a unique D such that $\sum_{i=1}^N r_i^D = 1$.

Definiton 5. *If $\sum_{i=1}^N r_i^D = 1$, then we say D is the **similarity dimension** of $|\mathcal{S}|$.*

The similarity dimension is often called the fractal dimension and was introduced by Mandelbrot (see [6]). An intuitive reasoning behind this definition is as follows. Often, a fractal exhibits a pattern that is invariant under scaling. That is, some portion of the fractal is a copy of the fractal as a whole. So the fractal doesn't lose its detail when repeatedly magnified. The similarity dimension measures the ratio of the change in detail of such a fractal to the change in scale. Note that the r_i are a measure of how $|\mathcal{S}|$ scales.

We will prove in section 7 that the similarity and Hausdorff dimensions of $|\mathcal{S}|$ are often equal.

5 Proof of existence and uniqueness of $|\mathcal{S}|$

We will prove the existence and uniqueness of $|\mathcal{S}|$ using the Banach fixed-point theorem, which states that a contraction mapping on a complete metric space admits a unique fixed-point [11, theorem 8.3.10]. First we need to construct the metric space we will use.

Definiton 6. *For $x \in \mathbb{R}^d$ and $E \subset \mathbb{R}^d$, the **distance** between x and A is $d(x, E) = \inf\{|x - y| : y \in E\}$.*

*Let $\mathcal{C}^d$ be the family of non-empty compact subsets of $\mathbb{R}^d$. Define the **Hausdorff metric** δ on $\mathcal{C}^d$ by*

$$\delta(E, F) = \max\left(\sup\{d(x, F) : x \in E\}, \sup\{d(y, E) : y \in F\}\right)$$

for $E, F \in \mathcal{C}^d$.

Lemma 2. *The Hausdorff metric δ is indeed a metric on $\mathcal{C}^d$ and, moreover, $(\mathcal{C}^d, \delta)$ is complete.*

The first part of this lemma is easy to verify. The second part follows from [4, result 2.10.21].

Lemma 3. *For $A_i, B_i \in \mathcal{C}^d$, we have*

$$\delta\left(\bigcup_{i\in I} A_i, \bigcup_{i\in I} B_i\right) \le \sup_{i\in I} \delta\left(A_i, B_i\right).$$

Proof. Without loss of generality, assume $\delta\left(\bigcup_{i\in I} A_i, \bigcup_{i\in I} B_i\right) = \sup\{d(x, \bigcup_{i\in I} B_i) : x \in \bigcup_{i\in I} A_i\}$. Clearly, for all $x \in A_j$, we have $d\left(x, \bigcup_{i\in I} B_i\right) \le d(x, B_j) \le \delta(A_j, B_j)$. Thus,

$$\delta\left(\bigcup_{i\in I} A_i, \bigcup_{i\in I} B_i\right) = \sup_{i\in I} \sup_{x\in A_i} d\left(x, \bigcup_{i\in I} B_i\right) \le \sup_{i\in I} \delta(A_i, B_i).$$

□

Lemma 4. *For $f : \mathbb{R}^d \to \mathbb{R}^d$ and $A, B \in \mathcal{C}^d$,*

$$\delta(f(A), f(B)) \le (\operatorname{Lip} f)\delta(A, B).$$

Proof. Firstly note that $f(A), f(B) \in \mathcal{C}^d$, so $\delta(f(A), f(B))$ is defined. Without loss of generality we can assume $\delta(f(A), f(B)) = \sup\{d(f(x), f(B)) : x \in A\}$. Now the proof is just a matter of expanding out definitions and using the Lipschitz property:

$$\begin{aligned}
\delta(f(A), f(B)) &= \sup\{\inf\{|f(x) - f(y)| : y \in B\} : x \in A\} \\
&\le \sup\{\inf\{(\operatorname{Lip} f)|x - y| : y \in B\} : x \in A\} \\
&= (\operatorname{Lip} f) \sup\{d(x, B) : x \in A\} \\
&\le (\operatorname{Lip} f)\delta(A, B).
\end{aligned}$$

□

Theorem 2. *A set of similitudes $\mathcal{S}$, seen as a function $\mathcal{C}^d \to \mathcal{C}^d$, is a contraction on $(\mathcal{C}^d, \delta)$.*

Proof. Firstly, note that $\mathcal{S}$ sends $\mathcal{C}^d$ to $\mathcal{C}^d$. We need to show that $\mathcal{S} : \mathcal{C}^d \to \mathcal{C}^d$ has Lipschitz constant less than 1. From the two lemmas above, we get that for any $E, F \in \mathcal{C}^d$,

$$\begin{aligned}\delta\left(\mathcal{S}(E), \mathcal{S}(F)\right) &= \delta\left(\bigcup_{i=1}^{n} S_i(E), \bigcup_{i=1}^{n} S_i(E)\right)\\ &\leq \max_{1\leq i\leq N} \delta(S_i(E), S_i(F))\\ &\leq \max_{1\leq i\leq N} r_i \delta(E, F).\end{aligned}$$

Since $r_i < 1$ for all $1 \leq i \leq N$, it follows that $\max_{1\leq i\leq N} r_i\delta(E,F) < \delta(E,F)$. Thus, $\operatorname{Lip}\mathcal{S} < 1$ as required. □

Theorem 3 (the Banach fixed point theorem)**.** *Let (X, d) be a complete non-empty metric space and $f : X \to X$ be a contraction. Then f admits a unique fixed-point. That is, there exists a unique $x \in X$ such that $f(x) = x$.*

This result, also known as the contraction mapping principle, is a standard result in analysis. A proof is given in [11, section 8.3]. The result that $|\mathcal{S}|$ exists and is unique now follows easily:

Proof of Theorem 1. Apply the Banach fixed point theorem to the contraction $\mathcal{S} : \mathcal{C}^d \to \mathcal{C}^d$. □

Moreover, the Banach fixed point theorem gives us a constructive method to find $|\mathcal{S}|$: start with an arbitrary set $E_0 \in \mathcal{C}^d$ and define a sequence $\{E_n\}$ by $E_n = \mathcal{S}(E_{n-1})$ for $n \geq 1$. Then $E_n \to |\mathcal{S}|$ with speed of convergence described by

$$\delta\left(|\mathcal{S}|, E_{n+1}\right) \leq (\operatorname{Lip}\mathcal{S})\,\delta\left(|\mathcal{S}|, E_n\right) \leq \left(\max_{1\leq i\leq N} r_i\right)\delta\left(|\mathcal{S}|, E_n\right).$$

6 Aside: the Scaling Property

In the introduction, we discussed using the notion of scaling as a way of informally defining the dimension of an object. In this section, we will formalise this property and show that the Hausdorff dimension satisfies it.

Let $C^d = \{(x_1, ..., x_d) \subset \mathbb{R}^d : 0 \leq x_i \leq 1\}$ be the unit cude in $\mathbb{R}^d$ and let $aC^d = \{(x_1, ..., x_d) \subset \mathbb{R}^d : 0 \leq x_i \leq a\}$ be C^d scaled by some factor $a > 0$. In the introduction, we said that aC^d is a^d copies of C^d. What we technically meant was that we could translate a^d copies of C^d so that the disjoint union of the resulting cubes was aC^d. The crucial property here is that aC^d 'occupies the same amount of space' as the a^d translated copies of C^d. In other words, $\lambda^d(aC^d) = a^d\lambda^d(C^d)$, where λ^d is the Lebesgue measure on $\mathbb{R}^d$.

More generally, let $f : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^{\geq 0}$ be a function that assigns a dimension $f(E)$ to subsets $E \subset \mathbb{R}^d$ and let μ_δ be a measure associated with each dimension $\delta \in \mathbb{R}^{\geq 0}$. We say f and μ_δ satisfy the scaling property if

$$\mu_{f(E)}(aE) = a^{f(E)}\mu_{f(E)}(E)$$

where aE is a scaling of $E \subset \mathbb{R}^d$ by a factor of $a > 0$.

Proposition 1. *The Hausdorff dimension with the associated k-dimensional Hausdorff measure satisfies the scaling property. In fact, for all $k \in \mathbb{R}^{\geq 0}$ and $a > 0$,*

$$\mathcal{H}^k(aE) = a^k\mathcal{H}^k(E).$$

Proof. The result follows straightforwardly from the fact that $\operatorname{diam}(aE_i) = a\operatorname{diam}(E_i)$. □

Unfortunately, we do not have an associated measure for the similarity dimension, so we cannot talk about whether it satisfies the scaling property in general. However, we will see that for sufficiently 'nice' subsets $E \subset \mathbb{R}^d$, we can construct a family of similitudes $\mathcal{S}$ such that $|\mathcal{S}| = E$ and the Hausdorff and similarity dimensions of $|\mathcal{S}|$ agree. Then, for such E, the scaling property is satisfied by the similarity dimension with the corresponding dimensional Hausdorff measures.

7 Agreement of the Hausdorff and Similarity Dimensions

In this section, we will show that the Hausdorff dimension equals the similarity dimension of $|\mathcal{S}|$, provided a certain separation condition holds:

Definiton 7. *The set of similitudes $\mathcal{S}$ satisfies the* ***open set condition*** *if there exists a non-empty open set O such that $S_i(O)$ are pairwise disjoint and contained in O:*

1. $\bigcup_{i=1}^{N} S_i(O) \subset O$, *and*
2. $S_i(O) \cap S_j(O) = \emptyset$ *if* $i \neq j$.

Theorem 4. *Suppose $\mathcal{S}$ satisfies the open set condition. Then the Hausdorff dimension of $|\mathcal{S}|$ equals the similarity dimension of $|\mathcal{S}|$.*

This theorem was first given in this form in [5], however an equivalent result was proven in [7]. We need to develop some more theory before we can prove it but first, we will apply it to an example.

8 Example: the Generalised Cantor Set

Let $d = 1$ and $\mathcal{S}_r = \{S_{1r}, S_{2r}\}$ where

$$S_{1r} : \mathbb{R} \to \mathbb{R} : x \mapsto rx,$$
$$S_{2r} : \mathbb{R} \to \mathbb{R} : x \mapsto r(x-1) + 1.$$

If $r = \frac{1}{3}$, then $\mathcal{S}_r(C) = C$ where C is the Cantor set. So $|\mathcal{S}_r| = C$. For $0 < r < \frac{1}{2}$, the invariant set $|\mathcal{S}_r|$ is called the generalised Cantor set. For a reference on the generalised cantor set see [2, chapter 1.7].

We can further consider the case $\frac{1}{2} < r < 1$. Here $\mathcal{S}_r([0,1]) = [0,1]$ so $|\mathcal{S}_r| = [0,1]$. The Hausdorff dimension of the closed interval $[0,1]$ is 1 (by a result in [9, p. 329]) but the similarity dimension is $D = -\log 2/\log r > 1$. Thus, the two notions of dimensions do not agree in this case. Intuitively, we need the open set condition so that $S_{1r}((0,1))$ and $S_{2r}((0,1))$ do not overlap. Otherwise, their intersection 'gets counted twice' in the calculation of the similarity dimension.

We will show explicitly that the open set condition doesn't hold: We will first show that any non-empty open interval (a,b) cannot satisfy the open set condition. Suppose (a,b) satisfied the open set condition. Then $S_{1r}(a,b) \subset (a,b)$ implies $a \leq 0$ and $S_{2r}(a,b) \subset (a,b)$ implies $b \geq 1$. Moreover, $S_{1r}(a,b) \cap S_{2r}(a,b) = \emptyset$ implies $br \leq ar + (1-r)$. Combining these facts we get

$$r \leq br \leq ar + (1-r) \leq (1-r),$$

which is a contradiction as $r > \frac{1}{2}$. The case for arbitrary non-empty open sets follows: Suppose $U \subset \mathbb{R}$ is a non-empty open set. By [3, theorem 4.6], U can be written as the countable union of disjoint non-empty intervals I_n. If U satisfies the open set condition, then so would each I_n.

In the case $0 < r < \frac{1}{2}$, the interval $(0,1)$ satisfies the open set condition. Thus, we can apply theorem 4 to calculate the Hausdorff dimension:

$$\dim|\mathcal{S}_r| = \frac{-\log 2}{\log r}.$$

For a reference on the Hausdorff dimension of the Cantor set and other fractals see [10, p. 146-156].

9 Densities and Invariant Measures

Definiton 8. *The **lower** k**-dimensional density** of the measure μ at the point $x \in \mathbb{R}^d$ is defined as:*

$$\theta^k_*(\mu, x) = \liminf_{\rho \to 0} \frac{\mu(B(x,\rho))}{\alpha_k \rho^k},$$

*Similarily, the **upper** k**-dimensional density** is:*

$$\theta^{k*}(\mu, x) = \limsup_{\rho \to 0} \frac{\mu(B(x,\rho))}{\alpha_k \rho^k}.$$

where $\alpha_k = \Gamma(\frac{1}{2})^k / \Gamma(\frac{k}{2}+1)$ is a normalising constant equal to the volume of the k-dimensional unit ball.

A result in [4, section 2.10] states that if $0 < \mu(A) < \infty$ and the upper density $\theta^{k*}(\mu, a)$ is uniformly bounded away from 0 and ∞ for all $a \in A$, then $0 < \mathcal{H}^k(A) < \infty$. In particular, A has Hausdorff dimension k. Thus, to prove theorem 4, we need only show that the upper D-dimensional density of μ is uniformly bounded away from 0 and ∞, when D is the similarity dimension of $|\mathcal{S}|$. But first we need to construct a suitable measure μ.

Definiton 9. *An exterior measure μ on $\mathbb{R}^d$ is **Borel regular** if all Borel sets are measurable and for every $A \subset X$ there exists a Borel $B \supset A$ with $\mu(A) = \mu(B)$.*

Definiton 10. *The **support** of an (exterior Borel regular) measure ν is:*

$$\mathrm{spt}\nu = \mathbb{R}^d - \{V : V \text{ open}, \nu(V) = 0\}.$$

Denote the set of finite Borel regular exterior measures with bounded support by $\mathcal{M}$:

$$\mathcal{M} = \{\mu : \mu(\mathbb{R}^d) < \infty \text{ and } \mu \text{ Borel regular with bounded support}\}.$$

Denote the subset of $\mathcal{M}$ of exterior measures μ with mass 1 by $\mathcal{M}^1$:

$$\mathcal{M}^1 = \{\mu \in \mathcal{M} : \mu(\mathbb{R}^d) = 1\}.$$

For the rest of this section, assume $\mu \in \mathcal{M}$.

Definiton 11. *Suppose $f : \mathbb{R}^d \to \mathbb{R}^d$ is continuous and sends bounded sets to bounded sets. Then the induced map $f_\# : \mathcal{M} \to \mathcal{M}$ sends μ to the pushforward of μ:*

$$f_\#\mu(E) = \mu(f^{-1}(E)).$$

Recall that $\mathcal{S} = \{S_1, ..., S_N\}$ is a finite set of similitudes with $\mathrm{Lip}S_i = r_i < 1$ and $\sum_{i=1}^N r_i^D = 1$.

Definiton 12. *Define $\mathcal{S}(\mu) = \sum_{i=1}^N r_i^D S_{i\#}\mu$ to be the convex combination of the pushforward measures $S_{i\#}\mu$ with coefficients r_i^D.*

Theorem 5. *There exists a unique $\mu \in \mathcal{M}^1$ that is invariant with respect to $\mathcal{S}$ in the sense that:*

$$\mathcal{S}(\mu) = \mu.$$

The proof of this theorem involves constructing a metric on $\mathcal{M}^1$, showing $\mathcal{S} : \mathcal{M}^1 \to \mathcal{M}^1$ is a contraction map in this metric space and applying Banach's fixed point theorem. The proof will be deferred to appendix A (section 11).

10 Proof of Hausdorff and Similarity Dimensions Agreement

In this section we prove theorem 4, which states that, under the open set condition, the Hausdorff and similarity dimensions agree.

First, we need some notation. Denote $S_i(S_j(A))$ by $S_{ij}(A)$ for $A \subset \mathbb{R}^d$. Similarly, denote $S_{i_1}(S_{i_2}(...(S_{i_p}(A)...)$ by $S_{i_1 \dots i_p}(A)$ for $A \subset \mathbb{R}^d$ and $i_1, ..., i_p \in \{1, ..., N\}$. For the rest of this section, let $i_1, ...i_p, ...$ be a sequence with each element $i_p \in \{1, ..., N\}$. Below are some technical lemmas needed for theorem 4. Their proofs are in appendix B (section 12).

Lemma 5. *For $A \subset \mathbb{R}^d$, we have the bound*

$$\operatorname{diam} S_{i_1 \dots i_p}(A) \leq r_{i_1} \cdot ... \cdot r_{i_p} \operatorname{diam} A.$$

Furthermore, if A is bounded then,

$$\operatorname{diam} S_{i_1 \dots i_p}(A) \to 0 \text{ as } p \to \infty.$$

Lemma 6. *We have the following inclusions:*

$$|\mathcal{S}| \supset S_{i_1}(|\mathcal{S}|) \supset ... \supset S_{i_1 \dots i_p}(|\mathcal{S}|) \supset ...$$

Moreover, $\bigcap_{i=1}^{\infty} S_{i_1 \dots i_p}(|\mathcal{S}|)$ is a singleton set whose member is denoted $x_{i_1 \dots i_p \dots}$. $|\mathcal{S}|$ is the union of these singletons.

Lemma 7. *Let O be the open set given by the open set condition and I be a finite set of tuples with elements in $\{1, ..., N\}$:*

$$I = \{(j_1, ...j_p, ..., j_M) : M \in \mathbb{N} \text{ and } j_p \in \{1, ..., N\}\}.$$

Suppose that for every sequence β with elements in $\{1, ..., N\}$, there exists exactly one $\alpha \in I$ such that α is an initial segment of β, that is:

$$\alpha = (j_1, ..., j_M) \text{ and } \beta = j_1, ..., j_M, j_{M+1}, ...$$

Then the family of sets $\{S_{j_1 \dots j_M}(O) : (j_1, ..., j_M) \in I\}$ is pairwise disjoint.

Lemma 8. *Suppose $0 < s_1 < s_2 < \infty$ and $0 < \rho < \infty$. Let $\{U_i\}$ be a family of disjoint open sets such that each U_i contains a ball of radius ρs_1 and is contained in a ball of radius ρs_2. Then, for any $x \in \mathbb{R}^d$, at most $(1 + 2s_2)^d s_1^{-d}$ of the $\overline{U}_i$ meet $B(x, \rho)$.*

Lemma 9. *Let O be the open set asserted to exist by the open set condition. We have the following inclusion:*

$$S_{i_1 \dots i_p}(|\mathcal{S}|) \subset \overline{S_{i_1 \dots i_p}(O)}.$$

Lemma 10. *If $\mu \in \mathcal{M}^1$ is the measure invariant with respect to $\mathcal{S}$, then the support of μ is $|\mathcal{S}|$:*

$$spt\mu = |\mathcal{S}|.$$

We can now prove the main result of this essay.

Proof of theorem 4. Let D be the similarity dimension of $|\mathcal{S}|$ and $\mu \in \mathcal{M}^1$ be the measure invariant with respect to $\mathcal{S}$. Assume without loss of generality that $r_1 \leq ... \leq r_N$.

Recall that it suffices to show that $\theta^{D*}(\mu, x)$ is uniformly bounded away from 0 and ∞ for all $x \in |\mathcal{S}|$. We will find constants λ_1 and λ_2 such that

$$0 < \lambda_1 \leq \theta^*_D(\mu, x) \leq \theta^{D*}(\mu, x) \leq \lambda_2 < \infty,$$

for all $x \in |\mathcal{S}|$.

First, determine λ_1: Given $x \in |\mathcal{S}|$, using lemma 6 we can write $x = x_{i_1 \dots i_p \dots}$ for some sequence $i_1, \dots, i_p, \dots$. Let $\rho > 0$ and consider $B(x, \rho)$. Using lemmas 5 and 6, there exists q such that $S_{i_1 \dots i_q}(|\mathcal{S}|) \subset B(x, \rho)$. Choose the least such q.

We have that $r_{i_1} \cdot \dots \cdot r_{i_q} \operatorname{diam}|\mathcal{S}| \geq \rho r_1$. (Recall that $|\mathcal{S}|$ is bounded so $\operatorname{diam}|\mathcal{S}|$ is finite.) Why? Suppose not. Then,

$$r_{i_1} \cdot \dots \cdot r_{i_{q-1}} \operatorname{diam}|\mathcal{S}| = \frac{r_{i_1} \cdot \dots \cdot r_{i_q}}{r_{i_q}} \operatorname{diam}|\mathcal{S}| \leq \frac{r_{i_1} \cdot \dots \cdot r_{i_q}}{r_1} \operatorname{diam}|\mathcal{S}| < \rho. \tag{1}$$

So $S_{i_1 \dots i_{q-1}}(|\mathcal{S}|) \subset B(x, \rho)$, which is a contradiction.

By lemma 10, $\mu(|\mathcal{S}|) = 1$. Now, note that

$$\mu(A) = \sum_{j=1}^{N} r_j^D S_{j\#}\mu(A) \geq r_k^D S_{k\#}\mu(A) = r_k^D \mu(S_k^{-1}(A)),$$

for any $k = 1, \dots, N$ and $A \subset \mathbb{R}^d$.

Applying this repeatedly to $A = S_{i_1 \dots i_q}(|\mathcal{S}|)$:

$$\begin{aligned}\mu(S_{i_1 \dots i_q}(|\mathcal{S}|)) &\geq r_{i_1}^D \mu(S_{i_1}^{-1}(S_{i_1 \dots i_q}(|\mathcal{S}|))) = r_{i_1}^D \mu(S_{i_2 \dots i_q}(|\mathcal{S}|)) \\ &\geq \dots \geq r_{i_1}^D \cdot \dots \cdot r_{i_q}^D \mu(|\mathcal{S}|) \\ &= r_{i_1}^D \cdot \dots \cdot r_{i_q}^D.\end{aligned}$$

Hence,

$$\frac{\mu(B(x,\rho))}{\alpha_D \rho^D} \geq \frac{\mu(S_{i_1 \dots i_q}(|\mathcal{S}|))}{\alpha_D \rho^D} \geq \frac{r_{i_1}^D \cdot \dots \cdot r_{i_q}^D}{\alpha_D \rho^D} \geq \frac{r_1^D}{\alpha_D (\operatorname{diam}|\mathcal{S}|)^D}.$$

Therefore, we have found our λ_1: we have $\theta_*^D(\mu, x) \geq r_1^D \alpha_D^{-1} (\operatorname{diam}|\mathcal{S}|)^{-D} > 0$ for all $x \in |\mathcal{S}|$.

We will now find λ_2. Again fix $\rho > 0$ and $x \in |\mathcal{S}|$. Let O be the open set given by the open set condition and suppose O contains a ball of radius s_1 and is contained in a ball of radius s_2.

Consider a sequence $j_1, j_2, \dots, j_p, \dots$ such that each $j_q \in \{1, \dots, N\}$. Select the least q such that $r_{j_1} \cdot \dots \cdot r_{j_q} \leq \rho$. (We can do this as $r_i < 1$ for all $i = 1, \dots, N$.) Then by the analogous reasoning as above (see equation 1), $r_1 \rho \leq r_{j_1} \cdot \dots \cdot r_{j_q}$. Let I be the set of all such tuples $(j_1, \dots, j_q)$:

$$I = \{(j_1, \dots, j_q) : j_1, \dots, j_q \in \{1, \dots, N\} \text{ and } r_1 \rho \leq r_{j_1} \cdot \dots \cdot r_{j_q} \leq \rho\}.$$

Notice that I satisfies the conditions in lemma 7 so $\{S_{j_1 \dots j_q}(O) : (j_1, \dots, j_q) \in I\}$ is pairwise disjoint. Additionally, each $S_{j_1 \dots j_q}(O)$ contains a ball of radius $r_{j_1} \cdot \dots \cdot r_{j_q} s_1$ and consequently also contains a ball of radius $r_1 \rho s_1$. Similarly, each $S_{j_1, \dots, j_q}(O)$ is contained in a ball of radius $r_{j_1} \cdot \dots \cdot r_{j_q} s_2$ and consequently is also contained in a ball of radius ρs_2. Then by lemma 8, at most $(1 + 2s_2)^d (r_1 s_1)^{-d}$ elements of $\{\overline{S_{j_1 \dots j_q}(O)} : (j_1, \dots, j_q) \in I\}$ meet $B(x, \rho)$. It follows from lemma 9 that at most $(1 + 2s_2)^d (r_1 s_1)^{-d}$ elements of $\{S_{j_1 \dots j_q}(|\mathcal{S}|) : (j_1, \dots, j_q) \in I\}$ meet $B(x, \rho)$.

A result in [5, section 4.5] states, for $(j_1, \dots, j_q) \in I$,

$$\mu = \sum_{(j_1, \dots, j_q) \in I} r_{j_1}^D \cdot \dots \cdot r_{j_q}^D S_{j_1 \dots j_q \#}\mu \leq \sum_{(j_1, \dots, j_q) \in I} \rho^D S_{j_1 \dots j_q \#}\mu.$$

(The inequality follows easily from $r_{j_1} \cdot \dots \cdot r_{j_q} \leq \rho$.) Using lemma 10, the support of $S_{j_1 \dots j_q \#}\mu$ is $S_{j_1 \dots j_q}(|\mathcal{S}|)$. Therefore,

$$\mu(B(x, \rho)) \leq \sum_{(j_1, \dots, j_q) \in I} \rho^D S_{j_1 \dots j_q \#}\mu(B(x, \rho) \cap S_{j_1 \dots j_q}(|\mathcal{S}|)).$$

At most $(1 + 2s_2)^d (r_1 s_1)^{-d}$ of these summands are non-zero. We know $S_{j_1 \dots j_q \#}\mu(B(x, \rho) \cap S_{j_1 \dots j_q}(|\mathcal{S}|)) \leq S_{j_1 \dots j_q \#}\mu(\mathbb{R}^d) \leq 1$ since $\mu \in \mathcal{M}^1$. Hence,

$$\frac{\mu(B(x, \rho))}{\alpha_D \rho^D} \leq \frac{1}{\alpha_D \rho^D} \cdot \rho^D \cdot \frac{(1 + 2s_2)^d}{r_1^d s_1^d} = \frac{(1 + 2s_2)^d}{\alpha_D r_1^d s_1^d}.$$

Thus, we have found our λ_2: we have that $\theta^{D*}(\mu, x) \leq (1+2s_2)^d(\alpha_D r_1^d s_1^d)^{-1}$ for all $x \in |\mathcal{S}|$.

$\square$

11 Appendix A - the L metric on $\mathcal{M}^1$

For a reference see [1, section 2.6].

Definiton 13. *Define the L metric on $\mu, \nu \in \mathcal{M}^1$ by:*

$$L(\mu,\nu) = \sup\left\{\int \varphi d\mu - \int \varphi d\nu \middle| \varphi : \mathbb{R}^d \to \mathbb{R} \text{ Lipschitz with } \operatorname{Lip}\varphi \leq 1\right\}.$$

Verifying that L is indeed a metric is straightforward, except for checking the condition $L(\mu,\nu) < \infty$ which we will do now. Since μ and ν have bounded support, we can suppose $\mathrm{spt}\mu \cup \mathrm{spt}\nu \subset B(a,\rho)$ for some $a \in \mathbb{R}^d$ and $\rho > 0$. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ with $\operatorname{Lip}\varphi \leq 1$. We have

$$\int \varphi d\mu - \int \varphi d\nu = \int (\varphi - \varphi(a) + \varphi(a))d\mu - \int (\varphi - \varphi(a) + \varphi(a))d\nu = \int (\varphi - \varphi(a))d\mu - \int (\varphi - \varphi(a))d\nu,$$

since $\int \varphi(a)d\mu = \int \varphi(a)d\nu = \varphi(a)$. As φ has Lipschitz constant at most one, $|\varphi(x) - \varphi(a)| \leq \rho$ for all $x \in B(a,\rho)$. Hence

$$\int (\varphi - \varphi(a))d\mu - \int (\varphi - \varphi(a))d\nu \leq \int \rho d\mu + \int \rho d\nu = 2\rho.$$

So we have a bound on $L(\mu,\nu)$.

Proposition 2. *The metric L is complete.*

This proposition follows from [5, section 4.3].

We use the metric L and Banach's fixed point theorem to prove that there is a unique invariant measure $\mu \in \mathcal{M}^1$ with respect to $\mathcal{S}$.

Proof of theorem 5. It suffices to show $\mathcal{S} : \mathcal{M}^1 \to \mathcal{M}^1$ is a contraction map in the L metric. Firstly, we need to verify that $\mathcal{S}(\mu) \in \mathcal{M}^1$ for $\mu \in \mathcal{M}^1$. Since $S_i \in \mathcal{S}$ is a similitude, $S_{i\#}\mu$ has bounded support. It is straightforward to check that $S_{i\#}\mu$ is Borel regular and $S_{i\#}\mu(\mathbb{R}^d) = 1$. Thus, $S_{i\#}\mu \in \mathcal{M}^1$ and hence $\mathcal{S}(\mu) = \sum_{i=1}^N r_i^D S_{i\#}\mu \in \mathcal{M}^1$.

To show $\mathcal{S} : \mathcal{M}^1 \to \mathcal{M}^1$ is a contraction map, let $\varphi : \mathbb{R}^d \to \mathbb{R}$ with $\operatorname{Lip}\varphi \leq 1$ and let $r = \max_{1 \leq i \leq N} r_i$. (Recall $r_i < 1$ is the Lipschitz constant of $S_i \in \mathcal{S}$.)

In [5, section 2.5], Hutchinson shows $\int \varphi d(S_{i\#}\mu) = \int \varphi \circ S_i d\mu$. Thus, for $\mu, \nu \in \mathcal{M}^1$,

$$\begin{aligned} L(\mathcal{S}(\mu), \mathcal{S}(\nu)) = \int \varphi d(\mathcal{S}(\mu)) - \int \varphi d(\mathcal{S}(\nu)) &= \int \varphi d\left(\sum_{i=1}^N r_i^D S_{i\#}\mu\right) - \int \varphi d\left(\sum_{i=1}^N r_i^D S_{i\#}\nu\right) \\ &= \sum_{i=1}^N r_i^D \left(\int (\varphi \circ S_i)d\mu - \int (\varphi \circ S_i)d\nu\right) \\ &= \sum_{i=1}^N r_i^D r \left(\int (r^{-1}\varphi \circ S_i)d\mu - \int (r^{-1}\varphi \circ S_i)d\nu\right). \end{aligned}$$

Realise that $r^{-1}\varphi \circ S_i$ is Lipschitz with Lipschitz constant at most $r^{-1} \cdot 1 \cdot r_i \leq 1$. Hence,

$$\int (r^{-1}\varphi \circ S_i)d\mu - \int (r^{-1}\varphi \circ S_i)d\nu \leq L(\mu,\nu).$$

Therefore, $L(\mathcal{S}(\mu), \mathcal{S}(\nu)) \leq \sum_{i=1}^N r_i^D r L(\mu,\nu) = rL(\mu,\nu)$. $\square$

12 Appendix B - Proofs of Lemmata in Section 10

Proof of lemma 5. The first part follows easily from the fact that S_i has Lipschitz constant r_i:

$$\operatorname{diam} S_i(A) = \sup\{|S_i(x) - S_i(y)| : x, y \in A\} \leq \sup\{r_i|x-y| : x, y \in A\} = r_i \operatorname{diam} A.$$

The second part follows from the first part and the fact $r_i < 1$ for all $i = 1, ..., N$. □

Proof of lemma 6. Using the invariance of $|\mathcal{S}|$:

$$|\mathcal{S}| = \bigcup_{i=1}^{N} S_i(|\mathcal{S}|) = \bigcup_{i,j} S_{ij}(|\mathcal{S}|) = ... = \bigcup_{i_1,...,i_p} S_{i_1...i_p}(|\mathcal{S}|).$$

Thus, $|\mathcal{S}| \supset S_{i_1}(|\mathcal{S}|) \supset ... \supset S_{i_1...i_p}(|\mathcal{S}|) \supset ....$ By lemma 5, $\operatorname{diam} S_{i_1...i_p}(|\mathcal{S}|) \to 0$ as $p \to \infty$. Therefore, $\bigcap_{p=1}^{\infty} S_{i_1...i_p}(|\mathcal{S}|)$ is a singleton set. □

Proof of lemma 7. Let $\gamma, \delta \in I$ and suppose $\gamma \neq \delta$. Consider the tuples $(j_1, ..., j_q)$ that are initial segments of both γ and δ. Chose the longest such tuple $(j_1, ..., j_q)$. (It is possible that $q = 0$.) By assumption there exist $j_{q+1} \neq j'_{q+1}$ such that $(j_1, ..., j_q, j_{q+1})$ is an initial segment of γ and $(j_1, ..., j_q, j'_{q+1})$ is an initial segment of δ. By 1. in the open set condition, $S_\gamma(O) \subset S_{j_1...j_q j_{q+1}}(O)$ and $S_\delta(O) \subset S_{j_1...j_q j'_{q+1}}(O)$. Hence, by 2. in the open set condition,

$$S_\gamma(O) \cap S_\delta(O) \subset S_{j_1...j_q}(S_{j_{q+1}}(O) \cap S_{j'_{q+1}}(O)) = \emptyset.$$

□

Proof of lemma 8. Suppose without loss of generality that $\overline{U}_1, ..., \overline{U}_m$ meet $B(0, \rho)$ and all other $\overline{U}_i$ do not. Then each $\overline{U}_1, ..., \overline{U}_m$ is a subset of $B(x, (1 + 2s_2)\rho)$. Hence, we have m disjoint balls of radius ρs_1 inside $B(x, (1 + 2s_2)\rho)$. The sum of their volumes must be less than the volume of $B(x, (1 + 2s_2)\rho)$:

$$m\alpha_d \rho^d s_1^d \leq \alpha_d (1 + 2s_2)^d \rho^d.$$

(Recall $\alpha_d = \Gamma(\frac{1}{2})^d / \Gamma(\frac{d}{2} + 1)$ is the volume of the unit ball.) This gives us the result. □

Proof of lemma 9. First note that $\overline{S_{i_1...i_p}(O)} = S_{i_1...i_p}(\overline{O})$. By the open set condition, $\mathcal{S}(\overline{O}) \subset \overline{O}$. Then $\overline{O} \supset \mathcal{S}(\overline{O}) \supset \mathcal{S}^2(\overline{O}) \supset ... \supset \mathcal{S}^p(\overline{O}) \supset ....$

Let $a \in \overline{O}$. By the above reasoning, $\lim_{p\to\infty} S_{i_1...i_p}(a) \in \overline{O}$. But realise that Banach's fixed point theorem applied to the contraction $\mathcal{S} : \mathcal{C}^d \to \mathcal{C}^d$ gives us a way of constructing $|\mathcal{S}|$ (see remark after proof of theorem 1). In particular, $\lim_{p\to\infty} S_{i_1...i_p}(a) = x_{i_1...i_p...}$. By lemma 6, $\mathcal{S}$ is the union of the $x_{i_1...i_p...}$. Thus $|\mathcal{S}| \subset \overline{O}$.

Finally, we get $S_{i_1...i_p}(|\mathcal{S}|) \subset \overline{S_{i_1...i_p}(O)}$ by applying $S_{i_1...i_p}$ to both sides of $|\mathcal{S}| \subset \overline{O}$. □

The proof of the final lemma of section 10 requires more theory on invariant measures and is given in [5, section 4.4(4)].

References

[1] F. J. Almgren, Jr., *Existence and regularity almost everywhere of solutions to elliptic variational problems among surfaces of varying topological type and singularity structure*, Annals of Mathematics **87** (1968), 321-391.

[2] D. Badziahin, A. Gorodnik and N. Peyerimhoff (ed.), *Dynamics and Analytic Number Theory*, London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 2016.

[3] N. L. Carothers, *Real Analysis*, Cambridge University Press, Cambridge, 2000.

[4] H. Federer, *Geometric Measure Theory*, Springer-Verlag, New York, 1969.

[5] J. E. Hutchinson, *Fractals and self similarity*, Indiana University Mathematics Journal **30** (1981), 713-747.

[6] B. Mandelbrot, *Fractals, Form, Chance and Dimension*, Freeman, San Francisco, 1977.

[7] P. A. P. Moran, *Additive functions of intervals and Hausdorff measure*, Mathematical Proceedings of the Cambridge Philosophical Society **42** (1946), 15-23.

[8] A. R. Pears, *Dimension Theory of General Spaces*, Cambridge University Press, Cambridge, 1975.

[9] E. M. Stein and R. Shakarchi, *Princeton Lectures in Analysis*, Vol. III, Princeton University Press, Princeton, 2003.

[10] S. Sternberg, *Dynamical Systems*, Dover Publications, New York, 2010.

[11] W. F. Trench, *Introduction to Real Analaysis*, Free Edition, Trinity University, San Antonio, 2003 (updated May 2012) (available online at `http://ramanujan.math.trinity.edu/wtrench/misc/index.shtml`).