Vector Fields on Spheres

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Contents

Notation

In the following, X, Y, A are topological spaces with $A \subset X$ having the subspace topology.

Notation

$*$	The space consisting of a single point.
I	The closed unit interval $[0, 1]$.
$X \coprod Y$	The disjoint union of X and Y.
X_+	The disjoint union of X and a point $*$.
S^n	The n-dimensional sphere, $S^n = \{v \in \mathbb{R}^{n+1} \mid \|\|v\|\| = 1\}$.
X/A	The quotient space of $X \coprod \{*\}$ by the equivalence relation $x \sim y$ if either $x, y \in A \coprod \{*\}$ or $x = y$. So $X/\emptyset = X_+$.
$X \vee Y$	The wedge sum of based spaces X and Y, defined by the quotient of $X \coprod Y$ where we have identified the basepoints of X and Y.
$\mathbb{N}$	$\{0,1,2,3,...\}$.
$\mathbb{N}_{>0}$	$\{1,2,3,...\}$.
$\mathbb{Z}$	The infinite cyclic group.
$\mathbb{Z}_n$	The cyclic group of order n.
$\mathbb{F}$	Denotes the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$.
$\mathbb{H}$	Denotes the quaternions.
$\mathbb{F}(n)$	The algebra of $n \times n$ matrices over $\mathbb{F}$.
$\mathbb{F}\mathrm{P}^n$	The n-dimensional $\mathbb{F}$-projective space, consisting of the lines passing through the origin in $\mathbb{F}^{n+1}$.
$\{e_1, ..., e_n\}$	The standard basis vectors of $\mathbb{F}^n$.

Top	The category of topological spaces.
$\mathbf{Top}_*$	The category of pointed topological spaces.
CW	The category of CW complexes.
$\mathbf{CW}_*$	The category of based CW complexes.
Ab	The category of abelian groups.
$W \cong Z$	Objects W and Z are isomorphic in the appropriate category. Usually denotes a homeomorphism between spaces, or weak equivalence of spectra.
$X \simeq Y$	Spaces X and Y are homotopy equivalent.
$[X, Y]$	The set of homotopy classes of maps $X \to Y$.
$K(G, n)$	The Eilenberg-MacLane space with $\pi_i K(G, n) \cong G$ if $i = n$ and trivial otherwise.

Chapter 1

Preliminaries

1.1 Introduction

We are concerned with determining the number of non-vanishing, linearly independent, tangent vector fields of S^n. In this thesis, we solve this problem largely by using tools of algebraic topology.

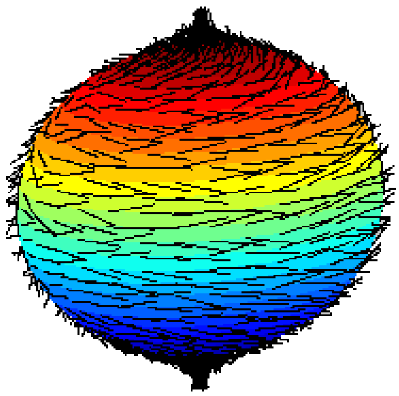


Figure 1.1: A hairy ball: a vector field on S^2 [26].

Definition 1.1. A *vector field* on S^{n-1} is a continuous map $v : S^{n-1} \to S^{n-1}$ such that $v(x)$ is tangent to x.

So we will require all vector fields herein to be non-vanishing, normalised and tangent.

Suppose $n = 2k$ for some integer k. Then S^{n-1} embeds in $\mathbb{C}^k$ and $v(x) = ix$ is a vector field. So we have at least one vector field on odd dimensional spheres. On the other, we can show easily that there are no vector fields on even dimensional spheres. If $v(x)$ was a vector field, then we would have a homotopy between the identity and the antipodal map:

$$h_t(x) = x \cos \pi t + v(x) \sin \pi t.$$

This contradicts the fact that the antipodal map has degree -1 on even dimensional spheres. This result is called the hairy ball theorem and was first proven by Brouwer [5]. In the case S^2, we can imagine the vector field as hairs on a ball. The theorem then states that it is not possible to comb a hairy ball without a cowlick.

Definition 1.2. For $n \in \mathbb{N}_{>0}$, write $n = (2a+1)2^b$ and $b = c + 4d$ for a, b, c, d integers and $0 \leq c \leq 3$. Define the *Radon-Hurwitz number* $\rho(n) = 2^c + 8d$.

n	1	2	3	4	5	6	7	8
$\rho(n)$	1	2	1	4	1	2	1	8

b	0	1	2	3	4	5	6	7
2^b	1	2	4	8	16	32	64	128
$\rho(n)$	1	2	4	8	9	10	12	16

The main result of this thesis is the following:

Theorem 1.3 [1, Theorem 1.1]**.** *The maximum number of linearly independent vector fields on S^{n-1} is exactly $\rho(n) - 1$.*

Interestingly, $\rho(n)$ only depends on the factors of two in n and therefore, grows surprisingly slowly.

The construction of $\rho(n) - 1$ vector fields on S^{n-1} was solved by Hurwitz, Radon and Eckmann, using real Clifford algebras [6]. The Radon-Hurwitz number arose from earlier work and has a number of applications in linear algebra [19]. Adams showed in 1962 that it was not possible to construct $\rho(n)$ vector fields [1].

Previous results proved the impossibility for $b \leq 3$ (Steenrod and Whitehead [21]) and $b \leq 10$ (Toda [22]). Adams' proof was the one of the first major use of a new powerful tool in algebraic topology at that time, called topological K theory. (The other being a greatly simplified proof of the Hopf invariant one theorem.) He utilised a reduction of the problem to one concerning real projective spaces, due to James:

Theorem 1.4 [13, Theorem 8.2]**.** *Suppose that $n - 1 \leq 2(n-k)$. There are k linearly independent vector fields on S^{n-1} if and only if the stunted projective space $\mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-k-1}$ is reducible: that is, there is a map*

$$f : S^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-k-1}$$

such that composition with the quotient map q

$$S^{n-1} \xrightarrow{f} \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-k-1} \xrightarrow{q} \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-2} = S^{n-1}$$

has degree 1.

The major work of [1] was Adams' computation of the K-theory of projective spaces along with the resulting corollary:

Theorem 1.5 [1, Theorem 1.2]**.** *The stunted projective space $\mathbb{R}\mathrm{P}^{n+\rho(n)}/\mathbb{R}\mathrm{P}^{n-1}$ is not coreducible, at least when n is a multiple of 16: that is, there is no map*

$$f : \mathbb{R}\mathrm{P}^{n+\rho(n)}/\mathbb{R}\mathrm{P}^{n-1} \to S^n$$

such that composition with the inclusion map i

$$S^n = \mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{n-1} \xrightarrow{i} \mathbb{R}\mathrm{P}^{n+\rho(n)}/\mathbb{R}\mathrm{P}^{n-1} \xrightarrow{f} S^n$$

has degree 1.

Theorem 1.3 had already been proven by Steenrod and Whitehead in [21] for the special case when n is not a multiple of 16. (We prove this special case in chapter 3.) Adams was then able to prove theorem 1.3 for n a multiple of 16 using theorem 1.5 and some stable homotopy theory.

In this thesis, we will provide the background to prove this result, assuming only some basic algebraic topology and category theory. To avoid pathologies, we assume that all spaces are compactly generated weak Hausdorff.

We begin in chapter 1 by outlining the necessary background. In chapter 2, we construct the $\rho(n) - 1$ vector fields on S^{n-1}. In chapter 3, we prove the special case of the main theorem (1.3), when n is not a multiple of 16. In chapter 4, we introduce spectra and define the stable homotopy category. In chapter 5, we present a number of results on duality which are needed to solve our vector field problem. In chapter 6, we develop spectral sequences and prove the existence of the Atiyah Hirzebruch spectral sequence. In chapter 7, we apply this spectral sequence to compute the K theory of stunted projective spaces. Finally, in chapter 8, we achieve our goal, by proving the two main theorems (1.5 and 1.3).

1.2 Some Basic Algebraic Topology

1.2.1 Join

Definition 1.6. The *join* of two spaces X and Y is

$$X * Y = X \times Y \times I/R,$$

where R is the relation defined by $(x, y, 0) \sim (x, y', 0)$ and $(x, y, 1) \sim (x', y, 1)$ for all $x, x' \in X$ and $y, y' \in Y$.

Intuitively, the join of X and Y is the space of lines connecting each point of X to each point of Y.

1.2.2 Smash Product

Definition 1.7. The *smash product* of two based spaces X and Y is the quotient of their cartesian product by their wedge sum:

$$X \wedge Y = (X \times Y)/(X \vee Y).$$

The sphere S^{n+m} is homeomorphic to the smash product $S^n \wedge S^m$.

Given based maps $f_1 : X_1 \to Y_1, f_2 : X_2 \to Y_2$, we have a map

$$\begin{aligned} f_1 \wedge f_2 : X_1 \wedge X_2 &\to Y_1 \wedge Y_2, \\ (x_1, x_2) &\mapsto (f_1(x_1), f_2(x_2)). \end{aligned}$$

The reduced suspension

Definition 1.8. The reduced suspension Σ is a functor from $\mathbf{Top}_*$ to $\mathbf{Top}_*$ defined on objects by

$$\Sigma X = S^1 \wedge X$$

and on maps $f : X \to Y$ by

$$\begin{aligned} \Sigma X &\to \Sigma Y, \\ \Sigma f(s, x) &\mapsto \big(s, f(x)\big). \end{aligned}$$

The k-th fold reduced suspension is homeomorphic to the smash product with S^k:

$$\Sigma^k X \cong S^k \wedge X.$$

The smash product gives a symmetric monoidal structure on the category of locally compact Hausdorff spaces. In this category, the internel hom exists and so we have an adjunction

$$\mathrm{Hom}(X \wedge A, Y) \cong \mathrm{Hom}\big(X, \mathrm{Hom}(A, Y)\big).$$

In particular, taking $A = S^1$, we recover the suspension-loopspace adjunction

$$\mathrm{Hom}(\Sigma X, Y) \cong \mathrm{Hom}(X, \Omega Y).$$

1.3 Bundles and K-Theory

1.3.1 Bundles

Vector Bundles

Intuitively, vector bundles are spaces that locally look like the cartesian product of a base space B and a vector space.

Definition 1.9. A (real) *vector bundle* consists of

1. topological spaces B and E,
2. a continuous surjection $p : E \to B$,
3. and a (real) finite vector space structure on $p^{-1}(b)$, for each $b \in B$,

satisfying the local trivialisation condition: there is an open covering $\{U_\alpha\}$ of B such that for each U_α there exists a positive integer n and a homeomorphism $h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ taking $p^{-1}(b)$ to $\{b\} \times \mathbb{R}^n$ by linear isomorphism for all $b \in U_\alpha$.

We call B the *base space*, E the *total space*, $p^{-1}(b)$ the *fibre* at $b \in B$ and h_α a *local trivialisation*.

A complex vector bundle is define analogously. Often only the total space E is given and the rest of the structure is left implicit.

Note that the dimension of the vector space structure on each fibre $p^{-1}(b)$ is constant in each U_α. It follows that the dimension is constant in each connected component of B. If the dimension n of $p^{-1}(b)$ is constant for all $b \in B$, then we call E an n-dimensional vector bundle.

A bundle that is 1-dimensional is called line bundle.

Definition 1.10. A *map* of vector bundles $E_1 \xrightarrow{p_1} B_1$ and $E_2 \xrightarrow{p_2} B_2$ is a continuous function $f : E_1 \to E_2$ which sends fibres to fibres by a linear transformation.

A vector bundle map $f : E_1 \to E_2$ is an *isomorphism* if it is a homeomorphism and a linear isomorphism on fibres.

Examples

Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. The n-dimensional trivial bundle $\epsilon^n \to X$ is the cartesian product $\mathbb{F}^n \times X$ with the projection map $p : (v, x) \mapsto x$.

The quotient space of

$$I \times \mathbb{R} \Big/ (0, v) \sim (1, -v)$$

is a real 1-dimensional bundle over S^1. It is called the Möbius bundle since it is homeomorphic to the Möbius strip with its boundary removed.

Real projective spaces has a canonical line bundle

$$E = \{(l, v) \in \mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1} \mid v \in l\}$$

where the fibre of each line l consists of vectors in l. These line bundles will play a major part in this thesis, as we will show they generate the K theory of $\mathbb{R}\mathrm{P}^n$. Complex projective spaces $\mathbb{C}\mathrm{P}^n$ also have canonical line bundles, defined in an analogous way.

Transition Functions

Given two local trivialisations $h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ and $h_\beta : p^{-1}(U_\beta) \to U_\beta \times \mathbb{R}^n$ with non-empty intersection, the homeomorphism $h_\beta \circ h_\alpha^{-1} : \big(U_\alpha \cap U_\beta\big) \times \mathbb{R}^n \to \big(U_\alpha \cap U_\beta\big) \times \mathbb{R}^n$ describes how the two local trivialisations are glued together. Since h_α, h_β send $p^{-1}(b)$ to $\{b\} \times \mathbb{R}^n$, the gluing map $h_\beta \circ h_\alpha^{-1}$ sends (b, v) to $\big(b, g_{\alpha\beta}(b)v\big)$, where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n)$.

Definition 1.11. The maps $g_{\alpha\beta}$ are called the *transition functions* of the bundle E.

The transitions functions satisfy $g_{\alpha\beta}(b) = g_{\beta\alpha}(b)^{-1}$ and the cocycle condition:

$$g_{\gamma\alpha}(b) \circ g_{\beta\gamma}(b) \circ g_{\alpha\beta}(b) = I_n$$

for all $b \in U_\alpha \cap U_\beta \cap U_\gamma$.

As demonstrated by the following example, the transition functions completely describe a vector bundle.

Example 1.12. *Given an open cover* $\{U_\alpha\}$ *and functions* $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n)$ *defined on the intersections satisfying* $g_{\alpha\beta}(b) = g_{\beta\alpha}(b)^{-1}$ *and the cocycle condition, we can build a vector bundle by forming the pieces* $\{U_\alpha \times \mathbb{R}^n\}$ *and gluing them together on the overlaps via* $g_{\alpha\beta}$.

We can easily extend this example to define a vector bundle with non-constant dimension.

Sections

Definition 1.13. A *section* of a vector bundle $p : E \to B$ is a continuous map $s : B \to E$ such that $p \circ s = \mathsf{id}_B$.

There is always a canonical section, the zero section, which sends $b \in B$ to the zero vector in the fibre $p^{-1}(b)$.

Example 1.14. *Given a (real) differentiable manifold* M *of dimension* n*, denote the tangent at* x *by* T_xM*. Then the tangent bundle* $T_M = \{(x, v) \mid x \in M, v \in T_xM\}$ *is a vector bundle of dimension* n*.*

Fibre Bundles

Fibre bundles are generalisations of vector bundles, in the sense that locally a fibre bundle looks like the cartesian product of the base space and the fibre, but the fibre do not need to have a vector space structure.

Definition 1.15. A *fibre bundle* consists of

1. topological spaces E, B and F,
2. a continuous surjection $p : E \to B$,

satisfying the local trivialisation condition: there is an open covering $\{U_\alpha\}$ of B such that for each U_α there exists a homeomorphism $h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F$ taking $p^{-1}(b)$ to $\{b\} \times F$ for all $b \in U_\alpha$.

We denote a fibre bundle by $F \to E \xrightarrow{p} B$.

We also have the notion of transition functions which must satisfy $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ and the cocycle condition. However now $g_{\alpha\beta}$ maps into the group of homeomorphism of F. As is the case for vector bundles, the transition functions contain all the information of the fibre bundle.

Fibrations

Definition 1.16. A map $p : E \to B$ has the *homotopy lifting property* with respect to a space X if the following condition is satisfied: Suppose there is a homotopy f :

$X \times I \to B$ and a map $\tilde{f}_0 : X \to E$ lifting f_0. That is, $f_0 = p \circ \tilde{f}_0$. Then there exists a homotopy $\tilde{f} : X \times I \to E$ lifting f. That is, there exists $\tilde{f}$ making the diagram

$$\begin{tikzcd} X \arrow[r, "\tilde{f}_0"] \arrow[d, hook, "X\times\{0\}"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & B \end{tikzcd}$$

commute.

Definition 1.17. A map $p : E \to B$ is a *(Hurewicz) fibration* if it has the homotopy lifting property with respect to any space. It is a *Serre fibration* if it has the homotopy lifting property with respect to any disc D^n (or equivalently, any finite CW complex).

Any fibre bundle is a Serre fibration. If the base space is paracompact, then it is also a Hurewicz fibration. For proof of these statements, see [9, Proposition 4.48] [20, §2.7] respectively. The fibres $p^{-1}(b)$ of a path-connected (Serre) fibration are (weakly) homotopy equivalent [9, Proposition 4.61]. So we can think of fibrations as the homotopy versions of fibre bundles. Like fibre bundles, we can write a path-connected fibration as $F \to E \xrightarrow{p} B$, where F is 'the' fibre.

1.3.2 K-Theory

Let $\mathrm{Vect}_{\mathbb{F}}(X)$ be the set of isomorphism classes of $\mathbb{F}$-vector bundles over a fixed base space X. Since $E_1 \cong E_1'$ and $E_2 \cong E_2'$ implies $E_1 \oplus E_2 \cong E_1' \oplus E_2'$, the direct sum is well defined on $\mathrm{Vect}_{\mathbb{F}}(X)$. Moreover, we have that $E \oplus \epsilon^0 = E$ for all vector bundles E. Thus, the direct sum gives a commutative monoid structure on $\mathrm{Vect}_{\mathbb{F}}(X)$.

Operations on Vector Bundles

Basically any vector space operation can also be done on vector bundles, by un-gluing the fibres, applying the operation to each fibre and then re-gluing the fibres back together.

Definition 1.18. Given two vector bundles $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ over a base space X, define their *direct sum*

$$E_1 \oplus E_2 = \{(e_1, e_2) \in E_1 \times E_2 \mid p_1(e_1) = p_2(e_2)\}.$$

Definition 1.19. Given two vector bundles $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ over a base space X, define their *tensor product* $E_1 \otimes E_2$ be the disjoint union of the vector spaces $p_1^{-1}(x) \otimes p_2^{-1}(x)$.

Proposition 1.20. *A continuous map $f : E_1 \to E_2$ between two vector bundles over the same base space is an isomorphism if it takes each fibre $p_1^{-1}(b)$ to the corresponding fibre $p_2^{-1}(b)$ by linear isomorphism.*

This is Lemma 1.1 in [10]. We omit the proof.

Definition 1.21. Given a map $f : A \to B$ and a vector bundle $p : E \to B$, the *pullback* of E by f is the vector bundle

$$f^*(E) = \{(v, a) \in E \times A \mid p(v) = f(a)\}.$$

The pullback $f^*(E)$ fits into the commutative square

$$\begin{tikzcd} f^*(E) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] & E \arrow[d, "p"] \\ A \arrow[r, "f"] & B, \end{tikzcd}$$

where $\mathrm{pr}_1(v, a) = v$ and $\mathrm{pr}_2(v, a) = a$.

Note that $f^*(E)$ only depends on the isomorphism type of E.

The construction of the pullback defines a map

$$\begin{aligned} f^* : \mathrm{Vect}_{\mathbb{F}}(B) &\to \mathrm{Vect}_{\mathbb{F}}(A) \\ E &\mapsto f^*(E) \end{aligned}$$

for any map $f : A \to B$. This is well defined since $f^*(E_1) \cong f^*(E_2)$ if $E_1 \cong E_2$.

The Functor K

We are now in a position to define K-theory.

Definition 1.22. Given a topological space X, define the $\mathbb{F}$ K *theory* $K_{\mathbb{F}}(X)$ to be the Grothendieck completion of the commutative monoid $(\mathrm{Vect}_{\mathbb{F}}(X), \oplus)$. That is, $K_{\mathbb{F}}(X)$ consists of formal differences $E_1 - E_2$ of vector bundles E_1, E_2, with the equivalence relation $E_1 - E_2 \sim E_1' - E_2'$ if there exists E_3 such that

$$E_1 \oplus E_2' \oplus E_3 = E_1' \oplus E_2 \oplus E_3,$$

and addition given in the obvious way:

$$(E_1 - E_2) + (E_1' - E_2') = (E_1 \oplus E_2) - (E_1' \oplus E_2').$$

We generally only consider compact Hausdorff X. This simplifies $K_{\mathbb{F}}(X)$, by the following proposition:

Proposition 1.23. *If X is compact Hausdorff, then for every vector bundle $E \to X$, there is a vector bundle $E' \to X$ such that $E \oplus E'$ is the trivial bundle.*

We will omit the proof of this proposition. It can be found in [10, Proposition 1.4]. Using this proposition, we see that $E_1 - E_2 = E_1' - E_2'$ in $K_{\mathbb{F}}(X)$ if and only if

$$E_1 \oplus E_2' \oplus \epsilon^n = E_1' \oplus E_2 \oplus \epsilon^n,$$

for some n. Moreover, every class in $K_{\mathbb{F}}(X)$ can be represented by a difference $E - \epsilon^n$ since given $E_1 - E_2$, there is a bundle E_3 with $E_2 \oplus E_3 = \epsilon^n$. So we get

$$E_1 - E_2 = E_1 \oplus E_3 - E_2 \oplus E_3 = E - \epsilon^n.$$

The function $f : X \to Y$ induces a map on K-theory $f^* : K_{\mathbb{F}}(Y) \to K_{\mathbb{F}}(X)$ by sending $E_1 - E_2$ to $f^*(E_1) - f^*(E_2)$. This is well defined and a group homomorphism.

Definition 1.24. The contravariant functor $K_{\mathbb{F}}$ from the category of compact spaces to **Ab** sends objects X to $K(X)$ and sends a map $f : X \to Y$ to its pullback $f^* : K_{\mathbb{F}}(Y) \to K_{\mathbb{F}}(X)$.

We can define a commutative multiplication on $K_{\mathbb{F}}(X)$ by the formula

$$(E_1 - E_2)(E_1' - E_2') = (E_1 \otimes E_1' - E_1 \otimes E_2') + (E_2 \otimes E_1' - E_2 \otimes E_2').$$

It is straightforward to check that this is indeed well defined and $\epsilon^1 - \epsilon^0$ is the identity. Moreover, $f^* : K_{\mathbb{F}}(Y) \to K_{\mathbb{F}}(X)$ commutes with this multiplication, so it is a ring homomorphism.

Let X have basepoint x_0. Define the reduced K-theory $\widetilde{K}_{\mathbb{F}}(X)$ as the kernel of the map induced by the inclusion $i : * \hookrightarrow X$. Elements of $\widetilde{K}_{\mathbb{F}}(X)$ are $E - \epsilon^n$ where the fibre of E has dimension n at $i^*(*)$. Since $\widetilde{K}_{\mathbb{F}}(X)$ is an ideal, we can think of it as a non-unital ring.

The dimension of a vector bundle remains constant under any induced map f^*. So if $E_1 - E_2$ is in the kernel of i^*, then $f^*(E_1) - f^*(E_2)$ will as well. We therefore get a contravariant functor $\widetilde{K}_{\mathbb{F}}$ from based compact spaces to **Ab**.

Notation

We write $E \in K_{\mathbb{F}}(X)$ for the class represented by $E - \epsilon^0$. Given a non-negative integer k, write $kE \in K_{\mathbb{F}}(X)$ for the k-fold sum of E and E^k for the k-fold product of E. For negative k, take the k-fold sum or product of the additive or multiplicative inverse of E (assuming a multiplicative inverse of E exists).

Universal Property of K Theory

Since we constructed $K_{\mathbb{F}}(X)$ as the Grothendieck completion of the commutative monoid $(\mathrm{Vect}_{\mathbb{F}}(X), \oplus)$, we have the following universal property:

$K_{\mathbb{F}}(X)$ is the unique abelian group such that for all abelian groups A and homomorphisms (of monoids) $f : \mathrm{Vect}_{\mathbb{F}}(X) \to A$, there exists a unique map (of abelian groups) $\overline{f} : K_{\mathbb{F}}(X) \to A$ such that

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "i"] & A \\
K_{\mathbb{F}}(X) \arrow[ur, dashed, "\overline{f}"'] &
\end{tikzcd}$$

So, if we can construct a map from $\mathrm{Vect}_{\mathbb{F}}(X)$ to an abelian group A, then we get a group homomorphism from $K(X)$ to A for free. A could be a ring, however, in this case, this map is not necessarily a ring homomorphism.

The Complexification Map

Given an n-dimensional real vector bundle $p : E \to B$, we can construct an n-dimensional complex vector bundle $c(E)$ as follows. Take each fibre and tensor with $\mathbb{C}$

over $\mathbb{R}$. The local trivialisations remain the same since $p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ becomes

$$\mathbb{C} \otimes p^{-1}(U_\alpha) \to U_\alpha \times (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^n) \xrightarrow{\cong} U_\alpha \times \mathbb{C}^n.$$

Equivalently, $c(E)$ is constructed from $E \oplus E$ by defining scalar multiplication by i in each fibre $\mathbb{R}^n \oplus \mathbb{R}^n$ via the rule $i(x, y) = (-y, x)$.

Note that $c(E_1 \oplus E_2) \cong c(E_1) \oplus c(E_2)$. Then $E_1 - E_2 = E_1' - E_2'$ in $K_{\mathbb{R}}(X)$ implies $c(E_1) - c(E_2) = c(E_1') - c(E_2')$. Also, c distributes across addition in $K_{\mathbb{R}}(X)$. Therefore, we get a group homomorphism $c : K_{\mathbb{R}}(X) \to K_{\mathbb{C}}(X)$ sending $E_1 - E_2$ to $c(E_1) - c(E_2)$.

We claim that additionally c is a ring homomorphism. Why? Suppose that E_1 and E_2 are n and m dimensional vector bundles. Since multiplication in $K_{\mathbb{R}}(X)$ is defined using the tensor product, it is sufficient to show that $c(E_1 \otimes_{\mathbb{R}} E_2) \cong c(E_1) \otimes_{\mathbb{C}} c(E_2)$.

Now, it is easy to see that

$$(\mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{R}^m) \oplus (\mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{R}^m) \cong (\mathbb{R}^n \oplus \mathbb{R}^n) \otimes_{\mathbb{C}} (\mathbb{R}^m \oplus \mathbb{R}^m),$$

where the direct sums have complex vector space structures given by $i(x, y) = (-y, x)$. Note that the left hand side is a fibre of $c(E_1 \otimes E_2)$ and the right hand side is a fibre of $c(E_1) \otimes c(E_2)$. We can easily extend this vector space isomorphism to a continuous map $c(E_1 \otimes_{\mathbb{R}} E_2) \to c(E_1) \otimes_{\mathbb{C}} c(E_2)$. Then by Proposition 1.20, we have that this map is a vector bundle isomorphism.

Going in the opposite direction, we can construct a $2n$-dimensional real vector bundle $r(E)$ from an n-dimensional complex vector bundle E by forgetting about the complex structure on the fibres. This gives a group homomorphism $r : K_{\mathbb{C}}(X) \to K_{\mathbb{R}}(X)$ sending $E_1 - E_2$ to $r(E_1) - r(E_2)$. However, it does not preserve multiplication.

1.4 Thom Spaces

Suppose we want to define an inner product on a vector bundle. This is not simply a matter of choosing an inner product for each fibre, since these choices may not agree when we glue the fibres together. Fortunately, when X is compact Hausdorff, there is an inner product on any vector bundle $p : E \to B$ [10, Proposition 1.2].

Now suppose we have an inner product on a vector bundle $p : E \to B$. (If B is paracompact, it is always possible to have a metric on E.) Define the sphere bundle $S(E)$ to be the space of vectors with norm equal to 1. Similarly define the disk bundle. These are fibre bundles, provided E has constant dimension. There are also definitions of $S(E)$ and $D(E)$ which do not use a metric but we will not concern ourselves with this.

Definition 1.25. The *Thom space* of a vector bundle E is the quotient

$$\mathrm{Th}(E) = D(E)/S(E).$$

Sometimes we will write $\mathrm{Th}(B, E)$ to emphasis that E is a vector bundle over the base space B. Isomorphic bundles give homeomorphic Thom spaces.

Proposition 1.26. *If X is a compact, then* $\mathrm{Th}(E)$ *is homeomorphic to* E^+ *the one point compactification of* E.

Proof. The bundle E is homotopy equivalent to $D(E) - S(E)$ by using a homotopy equivalence between $\mathbb{F}^n$ and the open ball. But $D(E) - S(E)$ is a subspace of $\mathrm{Th}(E)$ whose complement is a point.

Therefore, we have a continuous bijective map $\mathrm{Th}(E) \to E^+$. Since E^+ is Hausdorff and $\mathrm{Th}(E)$ is compact, this map must be a homeomorphism. $\square$

Proposition 1.27. *For a vector bundle $p : E \to B$, we have a homeomorphism*

$$\mathrm{Th}(E \oplus \epsilon^n) \cong \begin{cases} \Sigma^n \mathrm{Th}(E) & \text{for } E \text{ real,} \\ \Sigma^{2n} \mathrm{Th}(E) & \text{for } E \text{ complex.} \end{cases}$$

Proof. We only consider the case for real vector bundles, since this will suffice for our needs. Moreover, we need only prove the proposition for $n = 1$, since larger n will follow by induction:

$$\mathrm{Th}(E \oplus \epsilon^n) = \mathrm{Th}(E \oplus \epsilon^{n-1} \oplus \epsilon^1) \cong \Sigma \mathrm{Th}(E \oplus \epsilon^{n-1}).$$

For $n = 1$, define a norm on $E \oplus \epsilon^1 = E \times \mathbb{R}$ by $|(v, w)| = \max\{|v|_E, |w|_{\epsilon^n}\}$. Then

$$D(E \oplus \epsilon^1) \cong D(E) \times [0, 1]$$

and

$$S(E \oplus \epsilon^1) \cong \partial D(E \oplus \epsilon^1) = \{0, 1\} \times D(E) \cup [0, 1] \times S(E).$$

Therefore,

$$\begin{aligned} \mathrm{Th}(E \oplus \epsilon^1) &\cong \frac{[0,1] \times D(E)}{\{0,1\} \times D(E) \cup [0,1] \times S(E)} \\ &\cong \frac{S^1 \times D(E)}{* \times D(E) \cup S^1 \times S(E)} \\ &\cong \frac{S^1 \times \mathrm{Th}(E)}{* \times \mathrm{Th}(E) \cup S^1 \times *} = \Sigma \mathrm{Th}(E). \end{aligned}$$

$\square$

1.5 Cohomology Theories

Recall that a CW pair (X, A) is a CW complex X with a sub-complex inclusion $A \hookrightarrow X$. Every CW pair (X, A), for A non-empty, is a good pair, in the sense that A is a non-empty closed subspace that is a deformation retract of some open neighbourhood in X.

Definition 1.28. A *reduced cohomology theory* is a sequence of contravariant functors $\widetilde{E}^n$ (for $n \in \mathbb{Z}$) from $\mathbf{CW}_*$ to $\mathbf{Ab}$, with natural transformations $\delta : \widetilde{E}^n(A) \to \widetilde{E}^{n+1}(X/A)$ for CW pairs (X, A) satisfying the following axioms:

1. Homotopy: if $f, g : X \to Y$ are homotopic then the induced maps $f^*, g^* : \widetilde{E}^n(Y) \to \widetilde{E}^n(X)$ are equal.

2. Exactness: for each CW pair (X, A) there is a long exact sequence

$$... \xrightarrow{\delta} \widetilde{E}^n(X/A) \xrightarrow{q^*} \widetilde{E}^n(X) \xrightarrow{i^*} \widetilde{E}^n(A) \xrightarrow{\delta} \widetilde{E}^{n+1}(X/A) \xrightarrow{q^*} ...$$

   where i is the inclusion map $A \to X$ and q is the quotient map $X \to X/A$.

3. Additivity: For a wedge sum $X = \vee_\alpha X_\alpha$ with inclusions $i_\alpha : X_\alpha \hookrightarrow X$, the product map

$$\prod_\alpha i_\alpha^* : \widetilde{E}^n(X) \xrightarrow{\cong} \prod_\alpha \widetilde{E}^n(X_\alpha)^\dagger$$

   is an isomorphism for all n.

Definition 1.29. An *unreduced cohomology theory* is a sequence of contravariant functors E^n (for $n \in \mathbb{Z}$) from pairs of CW complexes (X, A) (with X non-empty) to abelian groups, with $E^n(X, A) = \widetilde{E}^n(X/A)$ for some reduced cohomology theory $\widetilde{E}$.

In the other direction, one gets a reduced theory $\widetilde{E}$ from an unreduced theory E by setting $\widetilde{E}^n(X)$ to be the kernel of the map $E^n(X) \to E^n(x_0)$ induced by the inclusion $x_0 \to X$.

1.6 Degree

In the introduction, we saw that the degree of a map $f : S^n \to S^n$ will play an important role in the proof of the main theorem.

Definition 1.30. Let $f : S^n \to S^n$. The *degree* of f is the integer $[f] \in \pi_n(S^n) \cong \mathbb{Z}$. (See [9, Corollary 4.25] for a proof that $\pi_n(S^n) \cong \mathbb{Z}$.)

Using this non-standard definition, we get Hopf's fundamental result for free: the degree of two maps $f, g : S^n \to S^n$ are equal if and only if $f \simeq g$.

The following proposition tells us that this definition agrees with the usual definition, where the degree of $f : S^n \to S^n$ is the integer d in the induced map on cohomology $f^* : H^n(S^n) \cong \mathbb{Z} \to H^n(S^n) \cong \mathbb{Z} : \alpha \mapsto d\alpha$.

Proposition 1.31. *If $f : S^n \to S^n$ has degree d, then the induced map f^* on any reduced cohomology theory is multiplication by d.*

This result is important for our purposes as it describes how a degree d map behaves on K theory.

Proof. For $d \geq 1$, f is homotopic to

$$g_d : S^n \xrightarrow{\text{pinch}} \vee_d S^n \xrightarrow{\vee \mathsf{id}} S^n$$

since we know g_d has degree d. Thus, we need only prove that g_d^* is multiplication by d.

If $\widetilde{E}$ is any reduced cohomology theory, then g_d^* is the composite

$$\begin{array}{ccccc} \widetilde{E}^k(S^n) & \xrightarrow{(\vee \mathsf{id})^*} & \widetilde{E}^k(\vee_d S^n) \cong \prod_d \widetilde{E}^k(S^n) & \xrightarrow{(\text{pinch})^*} & \widetilde{E}^k(S^n) \\ x & \longmapsto & (x, ..., x) & & \\ & & (x_1, ..., x_d) & \longmapsto & \sum_i x_i. \end{array}$$

(Why is $(\text{pinch})^*(x_1, ..., x_d) = \sum_i x_i$? When $d = 1$, we know that the pinch map is homotopic to the identity. It follows that $(\text{pinch})^*(0, ..., 0, x_i, 0, ..., 0) = x_i$. There is only one homomorphism that satisfies this property for all i, so $(\text{pinch})^*(x_1, ..., x_d)$ must be $\sum_i x_i$.) Thus, the composition is multiplication by d.

For $d = 0$, f is null-homotopic, so f^* must be multiplication by $0^\dagger$.

For $d \leq -1$, f is homotopic to a reflection r composed with g_{-d}, since we know that r has degree -1. (By a reflection, we mean a map that fixes a great circle S^{n-1} and interchanges the corresponding hemispheres.)

We claim that r^* is multiplication by -1. The composition

$$h : S^n \xrightarrow{\text{pinch}} S^n \vee S^n \xrightarrow{r \vee \mathsf{id}} S^n$$

is null-homotopic, since $[r]$ is inverse to $[\mathsf{id}_{S^n}]$ in $\pi_n(S^n)$. Thus, the induced map h^* is multiplication by 0, which implies $(r \vee \mathsf{id})^*$ sends $x \in \widetilde{E}^k(S^n)$ to $(-x, x) \in \widetilde{E}^k(S^n) \times E^k(S^n)$. □

1.7 Stunted Projective Spaces

Real stunted projective spaces also plays an important role in the proof of the main theorem.

Recall that we construct $\mathbb{R}\mathrm{P}^n$ from $\mathbb{R}\mathrm{P}^{n-1}$ by attaching a single n-cell with a two-to-one attaching map. Why? Define the relation $x \sim y$ if x, y are antipodal points on S^{n-1}. The inclusion of the boundary $i : S^{n-1} \to D^n$ respects this relation. Thus, i induces a map $S^{n-1}/\sim \cong \mathbb{R}\mathrm{P}^{n-1} \to D^n/\sim \cong \mathbb{R}\mathrm{P}^n$. It is then possible to see that attaching D^n via the quotient map $S^{n-1} \to S^{n-1}/\sim = \mathbb{R}\mathrm{P}^{n-1}$ gives $\mathbb{R}\mathrm{P}^n$ and this map is two-to-one.

Therefore, we can view $\mathbb{R}\mathrm{P}^k$ as a subcomplex of $\mathbb{R}\mathrm{P}^{n+k}$ consisting of all the cells of dimension at most k.

Definition 1.32. Given n and k positive, the *real stunted projective space* $\mathbb{R}\mathrm{P}_k^{n+k}$ is the quotient of $\mathbb{R}\mathrm{P}^{n+k}$ by $\mathbb{R}\mathrm{P}^{k-1}$

$$\mathbb{R}\mathrm{P}_k^{n+k} = \mathbb{R}\mathrm{P}^{n+k}/\mathbb{R}\mathrm{P}^{k-1}.$$

† Why is the constant map multiplication by 0? It factors through the space consisting of a single point $*$, which has cohomology groups all 0, for all reduced (ordinary) cohomology theories.

An analogous definition can be made for complex projective spaces.

As a CW complex, $\mathbb{R}\mathrm{P}_k^{n+k}$ consists of one cell in dimensions 0 and k, $k+1$,...,$k+n$. This justifies our notation. Moreover, we can include $\mathbb{R}\mathrm{P}_k^m$ in $\mathbb{R}\mathrm{P}_k^n$ as a CW complex, for $m \leq n$.

It is well known that the cohomology ring $H^*(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$, where $|\alpha| = 1$. By viewing $\mathbb{R}\mathrm{P}_k^{n+k}$ as a subcomplex of $\mathbb{R}\mathrm{P}^{n+k}$, it follows easily that

$$\widetilde{H}^p(\mathbb{R}\mathrm{P}_k^{n+k}; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & \text{if } k \leq p \leq n+k, \\ 0 & \text{otherwise.} \end{cases}$$

A more general definition of stunted projective space:

Sometimes it will be convenient to have a more general definition of real stunted projective space.

Definition 1.33. Let n be non-negative and let ξ be the canonical line bundle on $\mathbb{R}\mathrm{P}^n$. Define $\mathbb{R}\mathrm{P}_k^{n+k}$ to be the Thom space of $k\xi$:

$$\mathbb{R}\mathrm{P}_k^{n+k} = \mathrm{Th}(\mathbb{R}\mathrm{P}^n, k\xi).$$

The next proposition shows that this new definition is indeed a generalisation of the previous definition.

Proposition 1.34. *For n and k positive, we have a homeomorphism $\mathbb{R}\mathrm{P}_k^{n+k} \cong \mathbb{R}\mathrm{P}^{n+k}/\mathbb{R}\mathrm{P}^{k-1}$, between the old and new definitions of stunted projective space. For n non-negative and $k = 0$, $\mathbb{R}\mathrm{P}_0^n = \mathbb{R}\mathrm{P}_+^n$.*

Proof. The second statement is straightforward:

$$\mathbb{R}\mathrm{P}_0^n = \mathrm{Th}(\mathbb{R}\mathrm{P}^n, \epsilon^0) = D(\epsilon^0)/S(\epsilon^0) = \mathbb{R}\mathrm{P}^n/\emptyset = \mathbb{R}\mathrm{P}_+^n.$$

(This statement holds in general: $\mathrm{Th}(X, \epsilon^0) = X_+$.)

To prove the first statement, recall that

$$k\xi = \left\{(l, v_1, ..., v_k) \in \mathbb{R}\mathrm{P}^{n-1} \times \mathbb{R}^n \times ... \times \mathbb{R}^n \mid v_1, ..., v_k \in l\right\}.$$

Specifying a point $(l, v_1, ..., v_k) \in k\xi$ is equivalent to giving a linear function $f : l \to \mathbb{R}^k$. Why? Let $b_l \in l \subset \mathbb{R}^{n+1}$ be the point in l whose first non-zero co-ordinate (using the standard basis on $\mathbb{R}^{n+1}$) is 1. Then we can write each $v_1, ..., v_k$ as a scalar multiple of b_l. Denote these scalars by $[v_1]_{\beta_l}, ..., [v_k]_{\beta_l}$. We can completely define f by

$$f(b_l) = ([v_1]_{\beta_l}, ..., [v_k]_{\beta_l}) \in \mathbb{R}^k.$$

Conversely, given $f : l \to \mathbb{R}^k$, with $f(b_l) = (c_1, ..., c_k)$ we get a point $(l, c_1 b_l, ..., c_k b_l)$ in $k\xi$.

Now, any linear function $f : l \subset \mathbb{R}^{n+1} \to \mathbb{R}^k$ determines a graph in $\mathbb{R}^{n+1} \times \mathbb{R}^k$. Consider the quotient space of $\mathbb{R}^{n+1} \times \mathbb{R}^k - \{0\}$ where we identify $v \sim cv$ for all non-zero $c \in \mathbb{R}$. This quotient space is $\mathbb{R}\mathrm{P}^{n+k}$. Furthermore, every point in the graph of f

in $\mathbb{R}^{n+1} \times \mathbb{R}^k$ is a scalar multiple of $(b_l, f(b_l))$. Thus, the graph of f determines a point in $\mathbb{R}\mathrm{P}^{n+k}$. The only elements of $\mathbb{R}\mathrm{P}^{n+k}$ that are not equal to such a graph are the lines in $\mathbb{R}^k$.

Therefore, $k\xi$ is isomorphic to a vector bundle $\mathbb{R}\mathrm{P}^{n+k} - \mathbb{R}\mathrm{P}^{k-1} \to \mathbb{R}\mathrm{P}^n$. Since $\mathbb{R}\mathrm{P}^n$ is compact, $\mathrm{Th}(\mathbb{R}\mathrm{P}^{n+k} - \mathbb{R}\mathrm{P}^{k-1})$ is the one point compactification $\mathbb{R}\mathrm{P}^{n+k}/\mathbb{R}\mathrm{P}^{k-1}$. $\square$

1.8 Reducibility and Coreducibility

We have seen the definition of (co)reducibility of $\mathbb{R}\mathrm{P}^n_{n-k}$ in the introduction. In this section, we will make these definitions more precise and we will state some equivalent definitions. Intuitively, an n-dimensional CW complex is reducible if its top cell splits off and it is coreducible if its bottom cell splits off.

1.8.1 Reducibility

Definition 1.35. Let X be a $(n-1)$-dimensional CW complex and suppose Y is constructed from X by attaching an n-cell via $f : S^{n-1} \to X$. (That is, $Y = X \cup_f e_n$). Let $q : Y \to S^n$ be the map collapsing X to a point. We call this n-cell the top cell of Y and we say that Y is *reducible* if there exists a map $g : S^n \to Y$ such that post-composition with q

$$S^n \xrightarrow{g} Y \xrightarrow{q} S^n$$

has degree 1.

Lemma 1.36. *The space $Y = X \cup_f e_n$ is reducible if and only if f is null-homotopic.*

Proof. The composite $S^n \xrightarrow{g} Y \xrightarrow{q} S^n$ has degree 1 if and only if the boundary of the n-cell deformation retracts to a point. But the boundary of the n-cell is exactly the attaching map f, so this is equivalent to saying that f is null-homotopic. $\square$

There are a number of equivalent definitions, which will prove useful throughout this thesis. We take Y to be $\mathbb{R}\mathrm{P}^n_{n-k}$, for $n > k \geq 0$, but these results hold generally as well:

1. There is a map $S^n \to \mathbb{R}\mathrm{P}^n_{n-k}$ such that the composition $S^n \to \mathbb{R}\mathrm{P}^n_{n-k} \xrightarrow{q} \mathbb{R}\mathrm{P}^n_{n-k}/\mathbb{R}\mathrm{P}^{n-1}_{n-k} \cong S^n$ has degree 1, or equivalently, is homotopic to the identity.

2. There is a homotopy equivalence $\mathbb{R}\mathrm{P}^n_{n-k} \simeq \mathbb{R}\mathrm{P}^{n-1}_{n-k} \vee S^n$ which is the identity on the subcomplex $\mathbb{R}\mathrm{P}^{n-1}_{n-k}$.

3. The top cell of $\mathbb{R}\mathrm{P}^n_{n-k}$ has a trivial attaching map.

4. The top cell of $\mathbb{R}\mathrm{P}^n$ has an attaching map factoring through the $(n-k-1)$-skeleton.

1.8.2 Coreducibility

Definition 1.37. Let Y be a CW complex such that there is a single cell in dimension n, and all cells (except perhaps the base point 0-cell) have dimension greater than n. We call this n-cell the bottom cell and define $i : S^n \hookrightarrow Y$ to be the inclusion map of this cell. Then Y is *coreducible* if there exists a map $f : Y \to S^n$ such that pre-composition with i

$$S^n \xrightarrow{i} Y \xrightarrow{f} S^n$$

has degree 1.

Analogous to reducibility, there are a number of equivalent definitions of coreducibility, stated in terms of $\mathbb{R}\mathrm{P}_k^{n+k}$, for n, k positive, but easily generalisable:

1. There is a map $f : \mathbb{R}\mathrm{P}_k^{n+k} \to S^k$ such that the composition $S^k \xrightarrow{i} \mathbb{R}\mathrm{P}_k^{n+k} \xrightarrow{f} S^k$ has degree 1, or equivalently, is homotopic to the identity.

2. There is a homotopy equivalence $\mathbb{R}\mathrm{P}_k^{n+k} \simeq \mathbb{R}\mathrm{P}_{k+1}^{n+k} \vee S^k$ which is the identity on the subcomplex $\mathbb{R}\mathrm{P}_{k+1}^{n+k}$.

3. For $k < m \leq n + k$, the attaching map of the m-cell of $\mathbb{R}\mathrm{P}_k^{n+k}$ factors through $\mathbb{R}\mathrm{P}_{k+1}^{n+k}$. (It follows that the $(k+1)$-dimension cell of $\mathbb{R}\mathrm{P}_k^{n+k}$ has a trivial attaching map.)

1.9 Stiefel Manifolds

Let $O(n)$ be the orthogonal group of degree n:

$$O(n) = \left\{ M \in \mathbb{R}(n) \mid M \text{ invertible with } M^{-1} = M^T \right\},$$

where $\mathbb{R}(n)$ is the algebra of $n \times n$ real matrices. There are fibre bundles

$$\begin{aligned} O(n-1) \to O(n) &\xrightarrow{p} S^{n-1} \\ A &\mapsto Ae_1, \end{aligned}$$

where e_1 is the first standard basis vector of $\mathbb{R}^n$.

The fibres of these bundles are

$$p^{-1}(e_1) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & B & \\ 0 & & & \end{pmatrix}$$

where $B \in O(n-1)$. By induction, $O(n)$ looks like a twisted version of $S^0 \times S^1 \times ... \times S^{n-1}$. We can generalise this construction to get twisted versions of $S^k \times S^{k+1} \times ... \times S^{n-1}$ for any $k < n$.

Definition 1.38. The *Stiefel manifold* $V_{n,k}$ is the space of orthonormal k-frames in $\mathbb{R}^n$. In other words, it is the space of k-tuples of orthonormal vectors in $\mathbb{R}^n$.

The Stiefel manifold $V_{n,k}$ is a subspace of the product of k copies of the sphere S^{n-1}. It is given the subspace topology. Therefore, it is compact, since the product of spheres is compact.

Example 1.39. *1.* $V_{n,1} = S^{n-1}$,

2. $V_{n,n} \cong O(n)$, *where the homeomorphism is given by*

$$(v_1, ..., v_n) \mapsto [v_1 ... v_n] \in \mathbb{R}(n).$$

We know that $[v_1...v_n] \in O(n)$ *since a matrix* M *is orthogonal if* $M^T M = MM^T = I$*, or in other words, the rows and columns are orthonormal vectors.*

There are fibre bundles

$$\begin{aligned} V_{n-1,k-1} \longrightarrow V_{n,k} &\to S^{n-1} \\ (v_1, ..., v_k) &\mapsto v_1 \end{aligned}$$

Then, by induction, $V_{n,k}$ looks like a twisted version of $S^{n-k} \times ... \times S^{n-1}$.

Lemma 1.40. *Projection of the first vector of the k-tuple gives a fibre bundle* $\epsilon_1 : V_{n,k} \to S^{n-1}$*. Then* S^{n-1} *has* $k-1$ *vector fields if and only if* $\epsilon_1 : V_{n,k} \to S^{n-1}$ *admits a section.*

Proof. If we have $k-1$ vector fields $v_1, ..., v_{k-1}$ then

$$\begin{aligned} S : S^{n-1} &\to V_{n,k} \\ x &\mapsto \big(x, v_1(x), ..., s_{k-1}(x)\big) \end{aligned}$$

is a section of ϵ_1. Alternatively, from a section $s(x) = (x, s_1(x), ..., s_{k-1}(x))$, we get $k-1$ vector fields $s_1, ..., s_{k-1}$. $\square$

This lemma shows the importance of Stiefel manifolds to our problem of vector fields. We care about the cases when $k = \rho(n)$ and $k = \rho(n) + 1$. Moreover, James' reduction of the problem (Theorem 1.4) follows by embedding a stunted projective space within a Stiefel manifold as a sub-CW complex. To do this, we must study the CW structure of Stiefel manifolds.

A CW Structure on the Stiefel Manifolds

This CW structure was first described in [24]. A good reference is [9, pages 294-302].

Recall that the special orthogonal group $SO(n)$ is the subgroup of $O(n)$ consisting of matrices with determinant one. We will determine a CW structure on $SO(n)$ and then use this to induce a CW structure on $V_{n,k}$.

First, define the map $\rho : \mathbb{R}\mathrm{P}^{n-1} \to SO(n)$ as follows. To each non-zero vector $v \in \mathbb{R}^n$, write $r(v) \in O(n)$ for the reflection across the subspace orthogonal to v. Since $r(v)$ is a reflection, it has determinant -1. Composing with $r(e_n)$ gives an element of $SO(n)$. Since $r(v)r(e_n)$ only depends on the line spanned by v, the map $\rho([v]) = r(v)r(e_n)$ is well defined.

We claim that ρ embeds $\mathbb{R}\mathrm{P}^{n-1}$ as a subspace of $SO(n)$. The map $[v] \to r(v)$ is an injection of $\mathbb{R}\mathrm{P}^{n-1}$ into $O(n)-SO(n)$, and composition with $r(e_n)$ is a homeomorphism $O(n) - SO(n) \to SO(n)$. Thus, ρ is injective and since $\mathbb{R}\mathrm{P}^{n-1}$ is compact Hausdorff, ρ is an embedding.

We can generalise ρ to a map from a product of projective spaces $\mathbb{R}\mathrm{P}^{i_1} \times ... \times \mathbb{R}\mathrm{P}^{i_m}$ to $SO(n)$ where each $i_j < n$. Let $I = (i_1, ..., i_m)$. Given $[v_j] \in \mathbb{R}\mathrm{P}^{i_j}$, consider it as an element of $\mathbb{R}\mathrm{P}^{n-1}$ and apply ρ. Then let $\rho_I(v_1, ..., v_m)$ be the composition $\rho(v_1)\cdots\rho(v_m)$.

Now the cells of $\mathbb{R}\mathrm{P}^I = \mathbb{R}\mathrm{P}^{i_1} \times ... \times \mathbb{R}\mathrm{P}^{i_m}$ are simply the product of cells in each of $\mathbb{R}\mathrm{P}^{i_j}$. Moreover, if $\phi^i : D^i \to \mathbb{R}\mathrm{P}^i$ is the standard characteristic map of the top cell of $\mathbb{R}\mathrm{P}^i$, the product $\phi^I : D^I = D^{i_1} \times ... \times D^{i_m} \to \mathbb{R}\mathrm{P}^I$ is a characteristic map of the top cell of $\mathbb{R}\mathrm{P}^I$. (Note $D^{i_1} \times ... \times D^{i_m}$ is a disk of dimension $i_1 + ... + i_m$.)

Let a (possibly empty) sequence $I = (i_1, ..., i_m)$ be admissable if $n > i_1 > ... > i_m > 0$. Hatcher shows that the set of maps $\rho^I\phi^I : D^I \to SO(n)$, with I admissable, are the characteristic maps of a CW structure on $SO(n)$ [9, Proposition 3D.1]. For non-empty I, the cells $e^I = e^{i_1}\cdots e^{i_m}$ are products (via the group operation in $SO(n)$) of the cells $e^i \subset \mathbb{R}\mathrm{P}^i \subset \mathbb{R}\mathrm{P}^{n-1}$ embedded in $SO(n)$ by ρ. Thus, $SO(n)$ consists of a single 0-cell, corresponding to the empty sequence, and one cell for each admissable sequence $I = (i_1, ..., i_m)$ of dimension $i_1 + ... + i_m$. Finally, Hatcher also shows that $\rho_{(n-1,n-2,...,1)} : \mathbb{R}\mathrm{P}^{n-1} \times ... \times \mathbb{R}\mathrm{P}^1 \to SO(n)$ is cellular — a fact that will be useful later in the following subsection.

We can understand this result in the following way. Since there is a fibre bundle $SO(n-1) \to SO(n) \xrightarrow{p} S^{n-1}$ with $p(A) = Ae_1$, we know $SO(n)$ can be considered as a twisted product $S^{n-1} \times S^{n-2} \times ... \times S^1$.

Then for each admissable sequence $n > i_1 > ... > i_m > 0$, we choose the top cells of $S^{i_1}, ..., S^{i_m}$ and the 0-cell of the other spheres. We combine these cells to get a cell of $S^{n-1} \times S^{n-2} \times ... \times S^1$. By ranging over all admissable sequences, we get all the cells of $S^{n-1} \times S^{n-2} \times ... \times S^1$. Given that $SO(n)$ is a twisted product of these spheres, it is intuitive that it has the same cells.

Now we will use this result to determine a CW structure on the Stiefel manifold. There is a natural projection $p : O(n) \to V_{n,k}$, sending $\alpha \in O(n)$ to the last k columns of α. Similarly, there is a projection $p' : SO(n) \to V_{n,k}$. If $k < n$, then given an orthonormal k-frame $(v_1, ..., v_k)$, we can always find vectors $w_1, ..., w_{n-k}$ such that the matrix with columns $w_1, ..., w_{n-k}, v_1, ..., v_k$ is special orthogonal. Thus, p' is surjective if $k < n$. In this case, $V_{n,k}$ can be viewed as the coset space $SO(n)/SO(n-k)$, with the quotient topology from $SO(n)$. This is the same as the topology previously defined on $V_{n,k}$, since p' is a surjection between compact Hausdorff spaces.

We claim that the cells of $V_{n,k}$ are the sets of the form $e^I SO(n-k) = e^{i_1}\cdots e^{i_m}SO(n-$

k) for $n > i_1 > ... > i_m \geq n-k$, along with the coset $SO(n-k)$ as the 0-cell of $V_{n,k}$. These sets are the unions of cells in $SO(n)$ and moreover, every cell in $SO(n)$ is contained in exactly one of these sets. This implies that $V_{n,k}$ is the disjoint union of these sets. The dimension of a cell of the form $e^I SO(n-k)$ is $i_1 + ... + i_m$. Then if a cell in $SO(n)$ is contained in $e^I SO(n-k)$, its boundary is contained in cells of lower dimensions. This implies that the boundary of our proposed cells of $V_{n,k}$ are contained in some proposed cells of lower dimension. Thus, we have a CW structure on $V_{n,k}$.

The takeaway here is that there is a CW structure on $V_{n,k}$ which consists of one zero cell, and a cell e^I for every tuple $I = (i_1, ..., i_m)$ with $n > i_1 > ... > i_m \geq n-k$ of dimension $i_1 + ... + i_m$.

1.9.1 James' Results on Stiefel Manifolds

We are now in a position to prove the results on Stiefel manifolds necessary for our vector fields problem. These results are due to James, proved in further generality in section 3 of [14] and section 8 of [13].

Proposition 1.41. *The CW structure on $V_{n,k}$ given above admits $\mathbb{R}\mathrm{P}^{n-1}_{n-k}$ as the $2(n-k)$-skeleton. Hence, $\mathbb{R}\mathrm{P}^{n-1}_{n-k}$ is the $(n-1)$-skeleton if $2k-1 \leq n$.*

Proof. Suppose e^I is a cell with $I = (i_1)$ of length 1. We showed in the previous subsection that e^I is homeomorphic to the image of the i_1-cell of $\mathbb{R}\mathrm{P}^{n-1}$ under the map

$$\mathbb{R}\mathrm{P}^{n-1} \xrightarrow{\rho} S(n) \xrightarrow{q} S(n)/SO(n-k) \cong V_{n,k}.$$

This map collapses $\mathbb{R}\mathrm{P}^{n-k-1}$ to a point, while every cell of $\mathbb{R}\mathrm{P}^{n-1}_{n-k}$ is mapped homeomorphically onto some e^I with I length 1. Thus, $\mathbb{R}\mathrm{P}^{n-1}_{n-k}$ is a subcomplex of $V_{n,k}$.

If e^I is a cell of $V_{n,k}$ with I of length at least 2, then e^I has dimension at least $2n-2k+1$. Thus, the cells e^I with I length 1, along with the zero cell, account for all of the $(2n-2k)$-skeleton. But each e^I with $I = (i_1)$ of length 1 is the image under $\rho_{(n-1)}$ of the i_1 cell of $\mathbb{R}\mathrm{P}^{n-1}_{n-k}$. Thus, $\mathbb{R}\mathrm{P}^{n-1}_{n-k}$ is the $2(n-k)$-skeleton.

If $2k-1 \leq n$, then $n-1 \leq 2(n-k)$. In this case, there are no cells in $V_{n,k}$ of dimension d for $2(n-k) \leq d \leq n-1$. Thus, $\mathbb{R}\mathrm{P}^{n-1}_{n-k}$ is the $(n-1)$-skeleton of $V_{n,k}$. $\square$

Let $\iota : \mathbb{R}\mathrm{P}^{n-1}_{n-k} \hookrightarrow V_{n,k}$ be the map including $\mathbb{R}\mathrm{P}^{n-1}_{n-k}$ as a subcomplex. Note that $\rho(l)$ fixes the hyperplane perpendicular to the plane spanned by l and e_n, while $\rho(l)$ is a rotation of the plane spanned by l and e_n. Let $\epsilon_1 : SO(n) \to S^{n-1}$ be evaluation at e_1. Then $\epsilon_1 \rho(l) = e_1$ for any line l perpendicular to e_1. Thus the diagram

$$\begin{tikzcd} & SO(n) \arrow[d, "\epsilon_1"] \\ \mathbb{R}\mathrm{P}^{n-1} \arrow[ur, "\rho"] \arrow[r, "q"] & \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-2} \cong S^{n-1} \end{tikzcd}$$

is commutative. It follows that

$$\begin{tikzcd} & V_{n,k} \arrow[d, "\epsilon_1"] \\ \mathbb{R}\mathrm{P}^{n-1}_{n-k} \arrow[ur, "\iota"] \arrow[r, "q"] & \mathbb{R}\mathrm{P}^{n-1}_{n-k}/\mathbb{R}\mathrm{P}^{n-2}_{n-k} \cong S^{n-1} \end{tikzcd}$$

also commutes.

Corollary 1.42. *Suppose that $n-1 \leq 2(n-k)$. There is a section of $V_{n,k} \to S^{n-1}$ if and only if $\mathbb{R}\mathrm{P}^{n-1}_{n-k}$ is reducible.*

Proof. "$\Leftarrow$": Suppose there is a map $f : S^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}_{n-k}$ as in the definition of reducibility. Then there is a homotopy $F : I \times S^{n-1} \to S^{n-1}$ between the identity and $qf = \epsilon_1 \iota f$:

$$F_0 = \epsilon_1 \iota f \text{ and } F_1 = \mathsf{id}_{S^{n-1}}.$$

Therefore, we have a commutative square

$$\begin{tikzcd} S^{n-1} \arrow[r, "\iota f"] \arrow[d, "\{0\}\times S^{n-1}"'] & V_{n,k} \arrow[d, "\epsilon_1"] \\ I \times S^{n-1} \arrow[r, "F"] & S^{n-1}. \end{tikzcd}$$

So we are in a position to apply the homotopy lifting property for ϵ_1. This property guarantees a homotopy $\tilde{F} : I \times S^{n-1} \to V_{n,k}$ lifting F. By restricting to $S^{n-1} \times \{1\}$, we get

$$\begin{tikzcd} & V_{n,k} \arrow[d, "\epsilon_1"] \\ S^{n-1} \arrow[ur, "\tilde{F}_1"] \arrow[r, "\mathsf{id}"] & S^{n-1}. \end{tikzcd}$$

So $\widetilde{F}_1$ is the desired section.

"$\Rightarrow$": Suppose there is a map $s : S^{n-1} \to V_{n,k}$ such that $\epsilon_1 s = \mathsf{id}_{S^{n-1}}$. We can always find a homotopic map f which maps into the $(n-1)$-skeleton of $V_{n,k}$. So $\epsilon_1 f \simeq \mathsf{id}_{S^{n-1}}$. But the $(n-1)$-skeleton is $\mathbb{R}\mathrm{P}^{n-1}_{n-k}$ by the previous proposition. Thus we can consider f as a map $S^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}_{n-k}$ such that

$$fq = f\iota\epsilon_1 \simeq \mathsf{id}_{S^{n-1}}.$$

$\square$

This corollary makes the connection between the existence of vector fields and reducibility of stunted projective spaces.

1.10 K Theory as a Cohomology Theory

In this section, we will define functors $\widetilde{K}^n_{\mathbb{F}} : \mathbf{CW}_* \to \mathbf{Ab}$ that form a reduced cohomology theory. We set the 0-th functor to be the reduced K theory already defined. Then, we will show that complex K theory is two periodic and real K theory eight periodic. This will lead to the natural definitions

$$\widetilde{K}^n_{\mathbb{C}}(X) = \begin{cases} \widetilde{K}_{\mathbb{C}}(X) & \text{if } n \text{ is even,} \\ \widetilde{K}_{\mathbb{C}}(\Sigma X) & \text{if } n \text{ is odd,} \end{cases}$$

and

$$\widetilde{K}^n_{\mathbb{R}}(X) = \widetilde{K}_{\mathbb{R}}(\Sigma^r X),$$

where r is the remainder of $-n$ divided by 8.

1.10.1 Bott Periodicity

Theorem 1.43 [10, Theorem 2.11]**.** *Complex K theory is 2-periodic:*

$$\widetilde{K}_{\mathbb{C}}(\Sigma^2 X) \cong \widetilde{K}_{\mathbb{C}}(X),$$

for all compact Hausdorff spaces X.

This is a deep result and we will omit its proof. A corollary, also found in [10], is the case when $X = S^n$:

Corollary 1.44. *The complex K theory of spheres is*

$$\widetilde{K}_{\mathbb{C}}(S^n) = \begin{cases} \mathbb{Z} & \text{if } n \text{ even,} \\ 0 & \text{if } n \text{ odd.} \end{cases}$$

Also, $\widetilde{K}_{\mathbb{C}}(S^2)$ is generated by $\eta - \epsilon^1$, where η be that canonical line bunlde on $\mathbb{C}\mathrm{P}^1 = S^2$.

Theorem 1.45 [4]**.** *Real K theory is 8-periodic:*

$$\widetilde{K}_{\mathbb{R}}(\Sigma^8 X) \cong \widetilde{K}_{\mathbb{R}}(X),$$

for all compact Hausdorff spaces X.

Again, we omit the proof of this result.

Proposition 1.46. *There is an isomorphism between the homotopy classes of the infinite orthogonal group O and the real K theory of spheres:*

$$\pi_n O \cong \widetilde{K}_{\mathbb{R}}(S^{n+1}).$$

Thus,

$$\begin{array}{ccccccccccc} n & \equiv & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \mod 8 \\ \widetilde{K}_{\mathbb{R}}(S^n) & \cong & \mathbb{Z} & \mathbb{Z}/2 & \mathbb{Z}/2 & 0 & \mathbb{Z} & 0 & 0 & 0 & \\ \pi_n O & \cong & \mathbb{Z}/2 & \mathbb{Z}/2 & 0 & \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z} & \end{array}$$

Proof sketch. We know that $\pi_n O \cong \pi_n O(i)$ for i sufficiently large by examining the long exact sequence associated with the fibre bundle. Moreover, $\pi_n \mathrm{GL}_i(\mathbb{R})$ is in bijection with $\mathrm{Vect}^i_{\mathbb{R}}(S^{n+1})$ [10, Proposition 1.11]. Since $GL_i(\mathbb{R})$ deformation retracts to $O(i)$ via the Gram-Schmidt orthogonalisation process, we have that $\pi_n \mathrm{GL}_i(\mathbb{R}) \cong \pi_n O(i)$.

On the other hand, a class $\alpha \in \widetilde{K}_{\mathbb{R}}(S^{n+1})$ is represented by $[E] - [\epsilon^i]$ where $E \in \mathrm{Vect}^i_{\mathbb{R}}(S^{n+1})$. This gives a bijection between $\widetilde{K}_{\mathbb{R}}(S^{n+1})$ and $\mathrm{Vect}^i_{\mathbb{R}}(S^{n+1})$. So there is a bijection $\pi_n O(i) \to \widetilde{K}_{\mathbb{R}}(S^{n+1})$. It turns out that it is in fact an isomorphism. □

1.11 Adams Operations

Theorem 1.47. *There exist ring homomorphism $\Psi^k_{\mathbb{F}} : K_{\mathbb{F}}(X) \to K_{\mathbb{F}}(X)$ defined for all compact Hausdorff spaces X and integers k, which satisfy*

1. *naturality: $\Psi^k_{\mathbb{F}} f^* = f^* \Psi^k_{\mathbb{F}}$,*
2. *$\Psi^k_{\mathbb{F}}(L) = L^k$, for L a line bundle†,*
3. *$\Psi^k_{\mathbb{F}} \circ \Psi^l_{\mathbb{F}} = \Psi^{kl}_{\mathbb{F}}$,*
4. *$\Psi^p_{\mathbb{F}}(\alpha) \equiv \alpha^p \mod p$, for p prime, and*
5. *commutativity of the square*

$$\begin{tikzcd}
K_{\mathbb{R}}(X) \arrow[r, "\Psi^k_{\mathbb{R}}"] \arrow[d, "c"] & K_{\mathbb{R}}(X) \arrow[d, "c"] \\
K_{\mathbb{C}}(X) \arrow[r, "\Psi^k_{\mathbb{C}}"] & K_{\mathbb{C}}(X).
\end{tikzcd}$$

The ring homomorphisms $\Psi^k_{\mathbb{F}}$ are called Adams operations. Their existence is a well known result, so we will omit the proof. See for example [10, Theorem 2.20] for the complex Adams operations and [1, Theorem 5.1] for the general case.

Corollary 1.48. *The Adams operations on even dimension spheres*

$$\Psi^k_{\mathbb{C}} : \widetilde{K}_{\mathbb{C}}(S^{2q}) \to \widetilde{K}_{\mathbb{C}}(S^{2q}), \qquad (\textit{for arbitrary } q)$$
$$\Psi^k_{\mathbb{R}} : \widetilde{K}_{\mathbb{R}}(S^{2q}) \to \widetilde{K}_{\mathbb{R}}(S^{2q}), \qquad (\textit{for } q \textit{ even})$$

are given by

$$\Psi^k_{\mathbb{F}}(\kappa) = k^q \kappa.$$

Proof. We first prove the complex case by induction on q. The real case will then follow. In the case $q = 1$, we know that

$$\widetilde{K}_{\mathbb{C}}(S^2) \cong \mathbb{Z}\{\eta - \epsilon^1\}$$

†Since line bundles are invertible, this also makes sense for $k < 0$.

where η is the canonical line bundle on $\mathbb{C}\mathrm{P}^1 = S^2$, and moreover $(\eta - \epsilon^1)^2 = 0$. Write 1 for ϵ^1, the multiplicative identity in $K_{\mathbb{F}}(X)$. Then

$$\Psi^k_{\mathbb{C}}(\eta - 1) = \Psi^k_{\mathbb{C}}(\eta) - \Psi^k_{\mathbb{C}}(1) = \eta^k - 1$$

by property 2 of Theorem 1.47. But

$$\eta^k - 1 = (\eta - 1 + 1)^k - 1 = 1 + k(\eta - 1) - 1 = k(\eta - 1).$$

since $(\eta - 1)^2 = 0$.

In the step case, we use Bott periodicity $\widetilde{K}_{\mathbb{C}}(S^{2q+2}) \cong \widetilde{K}_{\mathbb{C}}(S^2) \otimes \widetilde{K}_{\mathbb{C}}(S^{2q})$. Suppose $x \in \widetilde{K}_{\mathbb{C}}(S^{2q}) \cong \mathbb{Z}$ is a generator. Then $(\eta - 1) \otimes x$ generates $\widetilde{K}_{\mathbb{C}}(S^{2q+2}) \cong \mathbb{Z}$ and

$$\Psi^k_{\mathbb{C}}\left((\eta - 1) \otimes x\right) = k(\eta - 1) \otimes k^q x$$

by the induction hypothesis. Since $k(\eta - 1) \otimes k^q x = k^{q+1}(\eta - 1) \otimes x$, we have the result.

Recall the complexification map c and the injection r from subsection 1.3.2. To prove the real case, we first observe that rc sends a vector bundle E to $E \oplus E$, so

$$\mathbb{Z} \cong K_{\mathbb{R}}(S^{2q}) \xrightarrow{c} K_{\mathbb{C}}(S^{2q}) \xrightarrow{r} K_{\mathbb{R}}(S^{2q}) \cong \mathbb{Z}$$

is multiplication by 2. Then the complexification map c is non-zero and therefore it is multiplication by some non-zero integer. Since c commutes with the Adam's operations (property 5 of Theorem 1.47), the real case follows. $\square$

Chapter 2

Constructing Vector Fields

In this chapter, we will determine a lower bound on the number of (non-vanishing, linearly independent tangent) vector fields on the $(n-1)$-sphere S^{n-1}.

Definition 2.1. A k-field is an ordered set of k point-wise orthonormal vector fields. A k-frame is an orthonormal set of k vectors.

Using the Gram-Schmidt process, we can construct a k-field from an ordered set of k point-wise linearly independent vector fields. Therefore, our problem of finding k linearly independent vector fields is equivalent to finding a k-field.

Let $K(n)$ be the maximal k such that there exists a k-field on S^{n-1}. We will construct a lower bound on $K(n)$.

2.1 Clifford Algebras

Definition 2.2. Define C_k^+ to be the free, associative $\mathbb{R}$-algebra with generators $e_1, ..., e_k$ and relations

$$e_i^2 = -1 \text{ for all } i, \quad e_ie_j + e_je_i = 0 \text{ for all } i \neq j.$$

Define C_k^- to be the free, associative $\mathbb{R}$-algebra with generators $e_1, ..., e_k$ and relations

$$e_i^2 = 1 \text{ for all } i, \quad e_ie_j + e_je_i = 0 \text{ for all } i \neq j.$$

Proposition 2.3. *We have the following $\mathbb{R}$-algebra isomorphisms:*

1. $C_1^+ \cong \mathbb{C}$,
2. $C_2^+ \cong \mathbb{H}$,
3. $C_1^- \cong \mathbb{R} \oplus \mathbb{R}$,[†]

[†]The direct sum of algebras X and Y over a field F is their direct sum as vector spaces, with multiplication defined by

$$(x_1, y_1)(x_2, y_2) = (x_1x_2, y_1y_2).$$

Thus, $\mathbb{R} \oplus \mathbb{R}$ is an algebra over $\mathbb{R}$.

4. $C_2^- \cong \mathbb{R}(2)$,

where $F(n)$ is the algebra of $n \times n$ matrices over F.

Proof. 1. C_1^+ is 2-dimensional with $e_1^2 = -1$. An arbitrary element of C_1^+ can be written in the form $a + be_1$ for $a, b \in \mathbb{R}$. Thus, we have an isomorphism

$$\begin{aligned} C_1^+ &\xrightarrow{\cong} \mathbb{C}, \\ a + be_1 &\longmapsto a + bi. \end{aligned}$$

2. C_2^+ is 4-dimensional with basis $\{1, e_1, e_2, e_1e_2\}$. Define $\phi : C_2^+ \to \mathbb{H}$ on this basis

$$\phi(1) = 1, \quad \phi(e_1) = i, \quad \phi(e_2) = j, \quad \phi(e_1e_2) = k,$$

and extend linearly. To check that this is an algebra homomorphism, we compute:

$$\begin{aligned} \phi(e_1^2) = \phi(-1) = -1 = i^2 = \phi(e_1)\phi(e_1), \qquad & \phi(e_2^2) = \phi(-1) = -1 = j^2 = \phi(e_2)\phi(e_2), \\ \phi(e_1e_2) = k = ij = \phi(e_1)\phi(e_2), \qquad & \phi(e_2e_1) = \phi(-e_1e_2) = -k = -ij = ji = \phi(e_2)\phi(e_1). \end{aligned}$$

Note that all of the relations in $\mathbb{H}$ are satisfied in C_2^+ and ϕ is bijective.

3. Again, we can write an arbitrary element of C_1^- of the form $a+be_1$ where $a, b \in \mathbb{R}$ and $e_1^2 = 1$. Define

$$\begin{aligned} C_1^+ &\xrightarrow{\cong} \mathbb{R} \oplus \mathbb{R} \\ a + be_1 &\longmapsto (a + b, a - b) \end{aligned}$$

It is straightforward to check that this is an algebra isomorphism.

4. C_2^- is 4-dimensional with basis $\{1, e_1, e_2, e_1e_2\}$. Define $\phi : C_2^- \to \mathbb{R}(2)$ on this basis

$$\phi(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \phi(e_1) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \phi(e_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \phi(e_1e_2) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and extend linearly. To check that this is an algebra homomorphism, we compute:

$$\phi(e_1^2) = \phi(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \phi(e_1)\phi(e_1),$$

$$\phi(e_2^2) = \phi(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \phi(e_2)\phi(e_2),$$

$$\phi(e_1e_2) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \phi(e_1)\phi(e_2),$$

$$\phi(e_2e_1) = \phi(-e_1e_2) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \phi(e_2)\phi(e_1).$$

The homomorphism ϕ has trivial kernel. Also, ϕ is surjective since

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \phi\left(a + (d-a)e_1 + be_2 + (c-b)e_1e_2\right). \qquad \square$$

Theorem 2.4. *For all $k \geq 0$,*

1. $C^+_{k+2} \cong C^-_k \otimes C^+_2$,†
2. $C^-_{k+2} \cong C^+_k \otimes C^-_2$.

Proof. Let $\{e_i\}_{i=1}^{k+2}$ be a basis for C^+_{k+2}. Let $\{e'_i\}_{i=1}^{k}$ be the generators of C^-_k and $\{e''_1, e''_2\}$ the generators of C^+_2. Define an $\mathbb{R}$-module homomorphism $u : C^+_{k+2} \to C^-_k \otimes C^+_2$ by

$$u(e_i) = \begin{cases} 1 \otimes e''_i & \text{if } i = 1, 2, \\ e'_{i-2} \otimes e''_1 e''_2 & \text{otherwise,} \end{cases}$$

and extend linearly. It is straightforward to check that u obeys the relations of C^+_{k+2}:

$$u(e_i)^2 = -1 \text{ and } u(e_i)u'(e_j) = -u(e_j)u(e_i) \text{ for } i \neq j.$$

So u is well defined. This also shows that u is an algebra homomorphism.

We will show that u is an isomorphism. Notice that u sends distinct basis elements to distinct basis elements. It follows that u is injective. The dimensions of C^+_{k+2} and $C^-_k \otimes C^+_2$ are equal. Thus, u is an isomorphism.

The second half of the theorem follows similarly: define $v : \mathbb{R}^{k+2} \to C^+_k \otimes C^-_2$ by

$$v(e_i) = \begin{cases} 1 \otimes e''_i & \text{if } i = 1, 2, \\ e'_{i-2} \otimes e''_1 e''_2 & \text{otherwise,} \end{cases}$$

where $\{e_i\}_{i=1}^{k+2}$ is a basis for $\mathbb{R}^{k+2}$, $\{e'_i\}_{i=1}^{k}$ are the generators of C^+_k and $\{e''_1, e''_2\}$ the generators of C^-_2. By analogous working to the first half, we can show v extends to an isomorphism from C^-_{k+2}. $\square$

Corollary 2.5. *For all $k \geq 0$,*

1. $C^+_{k+4} \cong C^+_k \otimes C^+_4$,
2. $C^-_{k+4} \cong C^-_k \otimes C^-_4$.

Proof. Apply Theorem 2.4 twice:

$$C^+_{k+4} \cong C^-_{k+2} \otimes C^+_2 \cong C^+_k \otimes C^-_2 \otimes C^+_2 \cong C^+_k \otimes C^+_4.$$

The second case follows analogously. $\square$

†If X and Y are F-algebras then $X \otimes Y$ is the tensor product of X and Y as vector spaces with multiplication defined by

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = x_1x_2 \otimes y_1y_2,$$

and then extended linearly to all elements of $X \otimes Y$.

By inspection, we can determine

1. $C_1^+ \cong \mathbb{C}$ and $C_1^- \cong \mathbb{R} \oplus \mathbb{R}$,
2. $C_2^+ \cong \mathbb{H}$ and $C_2^- \cong \mathbb{R}(2)$.

Then the following corollary is immediate.

Corollary 2.6. $C_4^{\pm} = \mathbb{H} \otimes \mathbb{R}(2) \cong \mathbb{H}(2)$.

Corollary 2.7. *For all* $k \geq 0$*,*

1. $C_{k+8}^+ \cong C_k^+ \otimes \mathbb{R}(16)$,
2. $C_{k+8}^- \cong C_k^- \otimes \mathbb{R}(16)$.

Proof. Apply corollary 2.5 twice:

$$C_{k+8}^+ \cong C_{k+4}^+ \otimes C_4^+ \cong C_k^+ \otimes C_4^+ \otimes C_4^+.$$

Now use corollary 2.6:

$$C_k^+ \otimes C_4^+ \otimes C_4^+ \cong C_k^+ \otimes \mathbb{H} \otimes \mathbb{R}(2) \otimes \mathbb{H} \otimes \mathbb{R}(2) \cong C_k^+ \otimes \mathbb{R}(16).$$

The second case follows analogously. □

From these results we obtain the following table of Clifford algebras:

k	C_k^+	C_k^-
1	$\mathbb{C}$	$\mathbb{R} \oplus \mathbb{R}$
2	$\mathbb{H}$	$\mathbb{R}(2)$
3	$\mathbb{H} \otimes (\mathbb{R} \oplus \mathbb{R}) \cong \mathbb{H} \oplus \mathbb{H}$	$\mathbb{C} \otimes \mathbb{R}(2) \cong \mathbb{C}(2)$
4	$\mathbb{H} \otimes \mathbb{R}(2) \cong \mathbb{H}(2)$	$\mathbb{H} \otimes \mathbb{R}(2) \cong \mathbb{H}(2)$
5	$\mathbb{H}(2) \otimes \mathbb{C} \cong \mathbb{C}(4)$	$\mathbb{H}(2) \otimes (\mathbb{R} \oplus \mathbb{R}) \cong \mathbb{H}(2) \oplus \mathbb{H}(2)$
6	$\mathbb{H}(2) \otimes \mathbb{H} \cong \mathbb{R}(8)$	$\mathbb{H}(2) \otimes \mathbb{R}(2) \cong \mathbb{H}(4)$
7	$\mathbb{H}(2) \otimes (\mathbb{H} \oplus \mathbb{H}) \cong \mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{H}(2) \otimes \mathbb{C}(2) \cong \mathbb{C}(8)$
8	$\mathbb{R}(16)$	$\mathbb{R}(16)$

2.2 Constructing Vector Fields

Recall that given $n \in \mathbb{N}_{>0}$, we write $n = (2a+1)2^b$ and $b = c + 4d$ and define $\rho(n) = 8d + 2^c$. In this section, we will construct $\rho(n) - 1$ vector fields on S^{n-1}.

Proposition 2.8. *For an* n*-dimensional representation* V *of* C_k^+*,*

$$S(V) \simeq S^{n-1\dagger}$$

admits a k*-field. A* $(k+1)$*-frame for* $x \in S(V)$ *is* $\{x, e_1 x, ..., e_k x\}$*.*

†Given an arbitrary (non-metrisable) real vector space V, the sphere of V is the quotient subspace

$$S(V) := \left(V - \{0\}\right) \Big/ (x \sim \lambda x \text{ for all } \lambda > 0) \cdot$$

Proof. Firstly, redefine the inner product $\langle -|-\rangle$ on V so it is invariant under the C_k^+ action by averaging over the action. Now, verify that x is perpendicular to $e_i x$:

$$\langle x|e_i x\rangle = \langle e_i x|e_i e_i x\rangle = \langle e_i x| - x\rangle = -\langle x|e_i x\rangle,$$

which implies $\langle x|e_i x\rangle = 0$.

Next, verify that $e_i x$ is perpendicular to $e_j x$ for $i \neq j$:

$$\langle e_i x|e_j x\rangle = \langle e_i e_j e_i x|e_i e_j e_j x\rangle = \langle -e_j e_i^2 x, e_i e_j^2 x\rangle = \langle e_j x| - e_i x\rangle = -\langle e_i x|e_j x\rangle,$$

and so $\langle e_i x|e_j x\rangle = 0$ as required. This shows that $\{x, e_1 x, ..., e_k x\}$ is a $(k+1)$-frame. From this we build the the k-field easily. □

Proposition 2.9 (Periodicity). *If C_k^+ has an n-dimensional representation then C_{k+8}^+ has a $16n$-dimensional representation. Moreover, if a_k is the minimum dimension of a representation of C_k^+, then $a_{k+8} = 16a_k$.*

Proof. Notice that C_k^+ is a matrix algebra. C_{k+8}^+ is an algebra with matrices a factor of 16 larger than the size of the matrices in C_k^+. Thus, any module of C_{k+8}^+ must have dimension equal to the product of 16 and the dimension of a module of C_k^+. □

Proposition 2.10. *The minimum dimension of a representation of C_k^+ for $k = 0, ..., 8$ are given in the table below.*

k	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
C_k^+	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$
a_k	*1*	*2*	*4*	*4*	*8*	*8*	*8*	*8*	*16*

Corollary 2.11. *The minimum dimension a_k is a power of 2 for all k. Moreover, $\{a_k\}_{k\in\mathbb{N}}$ are all the powers of 2.*

Proof. Using Proposition 2.10, we can verify the first half for $k = 0, ..., 7$. Then, use the periodicity $a_{k+8} = 16a_k$ to prove for all k.

For the second half, note that

$$a_0 = 1, \quad a_1 = 2, \quad a_2 = 4, \quad a_4 = 8.$$

Thus, $2^{4a+b} = a_{8a+b}$ for $a, b \in \mathbb{N}$, with $0 \leq b \leq 3$. □

Lemma 2.12. *If S^{n-1} admits a k-field then S^{nq-1} admits a k-field, for all $q \in \mathbb{N}_{>0}$.*

Proof. Let $v_1, ..., v_k : S^{n-1} \to S^{n-1}$ be pointwise orthonormal vector fields. Consider S^{nq-1} as the join of q spheres S^{n-1}: choose some $\alpha_1, ..., \alpha_q \geq 0$ such that $\sum_{j=1}^q \alpha_j^2 = 1$. For $x \in S^{nq-1}$, write

$$x = (\alpha_1 x_1, ..., \alpha_q x_q),$$

where $x_j \in S^{n-1}$.

Define $v_i^* : S^{nq-1} \to \mathbb{R}^{nq-1}$ by

$$v_i^*(\alpha_1 x_1, ..., \alpha_q x_q) = (\alpha_1 v_i(x_1), ..., \alpha_q v_i(x_q)).$$

Compute

$$\langle x|v_i^*(x)\rangle = \sum_{j=1}^{q} \alpha_j^2 \langle x_j|v_i(x_j)\rangle = 0.$$

$$\langle v_h^*(x)|v_i^*(x)\rangle = \sum_{j=1}^{q} \alpha_j^2 \langle v_h(x_j)|v_i(x_j)\rangle = \begin{cases} \sum_{j=1}^{q} \alpha_j^2 = 1 & \text{if } h = i, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $v_1^*, ..., v_k^*$ is a k-field on S^{nq-1}. □

Theorem 2.13. *We have the following lower bound on the number of vector fields of S^{n-1}:*

$$K(n) \geq \rho(n) - 1.$$

Proof. Fix n. Write $n = (2a+1)2^b = (2a+1)2^c16^d$ for $0 \leq c \leq 3$. By Lemma 2.12, it suffices to show that $S^{2^b} - 1$ admits $\rho(n) - 1$ vector fields. Since $\rho(n) = \rho(2^b)$, this is equivalent to proving $K(2^b) \geq \rho(2^b) - 1$.

By Corollary 2.11, there exists some k such that $2^b = a_k$. Choose the maximal such k. Then $K(2^b) \geq k$. We will show that $k = \rho(2^b) - 1$.

Write $k = 8q + r$ where $0 \leq r \leq 7$. Then $a_k = 16^q a_r$ by Proposition 2.9. It follows $q = d$ and $a_r = 2^c$ since $a_r \leq 8$ by inspection of Proposition 2.10. Thus, $k = 8d + r$. Since we chose k maximal such that $a_k = 2^b$, we know that r is maximal such that $a_r = 2^c$. Again using Proposition 2.10, we get the following table which shows that $r = 2^c - 1$.

c	2^c	r
0	1	0
1	2	1
2	4	3
3	8	7

Thus, $k = 8d + 2^c - 1 = \rho(2^b) - 1$. □

Chapter 3

Steenrod Squares and the Proof of a Special Case

In this chapter, we prove the upper bound on vector fields for the special case where $n \not\equiv 0 \mod 16$:

Theorem 3.1. *Write $n = (2a+1)2^b$ and $b = c + 4d$. The sphere S^{n-1} cannot have $\rho(n)$ vector fields when $d = 0$.*

This was first done in [21] by Steenrod and Whitehead. To prove this case, they used properties of the Steenrod squares.

3.1 Steenrod Algebras and Squares

Steenrod squares are cohomology operations $\mathrm{Sq}^i : H^n(X;\mathbb{Z}_2) \to H^{n+i}(X;\mathbb{Z}_2)$, defined as the generators of a certain algebra, the mod 2 Steenrod algebra.

Definition 3.2. A *cohomology operation* is a natural transformation:

$$\Phi : H^i(-;A) \to H^j(-;B).$$

Theorem 3.3. *The cohomology operations $\Phi : H^i(-;A) \to H^j(-;B)$ are parametrised by*

$$H^j(K(A,i);B),$$

where $K(A,i)$ is the Eilenberg-MacLane space. (That is, there is a one to one correspondence between the cohomology operations and elements of $H^j(K(A,i);B)$.)

Proof sketch. We can define cohomology as $H^j(X;B) = [X, K(B,j)]$. So we can think of $\alpha \in H^j(K(A,i);B)$ as a map $K(A,i) \to K(B,j)$. Then define

$$\begin{aligned} \alpha^* : H^i(X;A) &\to H^j(X;B) \\ f &\mapsto \alpha \circ f \end{aligned}$$

by postcomposition with α. We omit the proof that this is well defined (that is, it doesn't depend on the choice of representative for α) and that it is natural.

Conversely, given $\Phi : H^i(-;A) \to H^j(-;B)$, we get

$$\Phi_{K(A,i)} : H^i(K(A,i);A) \to H^j(K(A,i);B),$$
$$\mathsf{id} \mapsto \alpha.$$

These two functions are inverses. $\square$

Example 3.4. *The cup square is a cohomology operation.*

Example 3.5. *The only cohomology operations $H^i(-;\mathbb{Q}) \to H^j(-;\mathbb{Q})$ come from iterated cup product. Why? It is possible to calculate $H^*(K(\mathbb{Q},i);\mathbb{Q})$, using spectral sequences, as the free graded-commutative algebra over $\mathbb{Q}$ with one generator. Then Theorem 3.3 tells us that all cohomology operations come from this algebra.*

Definition 3.6. A *stable* sequence of cohomology operations of degree d is a set of cohomology operations $\{\Phi_i : H^i(-;A) \to H^{i+d}(-;B)\}_{i\in\mathbb{Z}}$ that commutes with the suspension isomorphism: i.e. the square

$$\begin{tikzcd} H^i(X;A) \arrow[r, "\Phi_i"] \arrow[d, "\cong"] & H^{i+d}(X;B) \arrow[d, "\cong"] \\ H^{i+1}(\Sigma X;A) \arrow[r, "\Phi_{i+1}"] & H^{i+d+1}(\Sigma X;B) \end{tikzcd}$$

commutes.

Note that $\{\Phi_i\}_{i\in\mathbb{Z}}$ is often called, confusingly, *a* stable cohomology operation. We use the name 'sequence' to make apparent that it is a sequence of cohomology operations.

Example 3.7. *Given the short exact sequence $0 \to \mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$ where the map p is multiplication by p, we get a short exact sequence of chain complexes and therefore a long exact sequence of cohomology*

$$... \to H^i(X;\mathbb{Z}_p) \to H^i(X;\mathbb{Z}_{p^2}) \to H^i(X;\mathbb{Z}_p) \xrightarrow{\beta} H^{i+1}(X;\mathbb{Z}_p) \to ...$$

The boundary maps, called Bockstein homomorphisms,

$$\beta_p : H^i(-;\mathbb{Z}/p) \to H^{i+1}(-;\mathbb{Z}/p)$$

form a stable sequence of cohomology operations.

Example 3.8. *Using Example 3.5, we see that the only stable sequence of operations between cohomologies with rational coefficients is the zero sequence, since the cup square is not stable.*

Definition 3.9. The *mod p Steenrod algebra $\mathscr{A}_p$* is the $\mathbb{Z}/p$-algebra of stable cohomology operations from $H^*(-;\mathbb{Z}/p)$ to itself.

Multiplication of the algebra is defined to be function composition of the two sequences' components. Addition and scalar multiplication of the algebra are defined in the obvious way.

We will focus on the $p = 2$ case from here on.

Theorem 3.10. *For $p = 2$, $\mathscr{A}_p$ is generated by*

$$\mathrm{Sq}^i : H^*(-;\mathbb{Z}/2) \to H^{*+i}(-;\mathbb{Z}/2),$$

for $i \geq 1$, subject to the Adém relations:

$$\mathrm{Sq}^a \mathrm{Sq}^b = \sum_j \binom{b-j-1}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^j$$

for $a < 2b$.

Proof idea. Use the Serre spectral sequence to compute $H^*(K(\mathbb{Z}/2,n);\mathbb{Z}/2)$ by induction on n. $\square$

Example 3.11. *1.* $\mathrm{Sq}^0 = \mathsf{id}$,

2. $\mathrm{Sq}^1 \mathrm{Sq}^1 = 0$,

3. $\mathrm{Sq}^1 \mathrm{Sq}^2 = \mathrm{Sq}^3$,

4. $\mathrm{Sq}^2 \mathrm{Sq}^2 = \mathrm{Sq}^3 \mathrm{Sq}^1$,

5. $\mathrm{Sq}^2 \mathrm{Sq}^3 = \mathrm{Sq}^5 + \mathrm{Sq}^4 \mathrm{Sq}^1$.

Proposition 3.12. *The Steenrod squares $\mathrm{Sq}^i : H^n(X;\mathbb{Z}/2) \to H^{n+i}(X;\mathbb{Z}/2), i \geq 0$, satisfy the following list of properties:*

1. *naturality:* $\mathrm{Sq}^i\left(f^*(\alpha)\right) = f^*\left(\mathrm{Sq}^i(\alpha)\right)$ *for $f : X \to Y$,*
2. *additivity:* $\mathrm{Sq}^i(\alpha + \beta) = \mathrm{Sq}^i \alpha + \mathrm{Sq}^i \beta$,
3. *the Cartan formula:* $\mathrm{Sq}^i(\alpha \smile \beta) = \sum_j \mathrm{Sq}^j \alpha \smile \mathrm{Sq}^{i-j} \beta$,
4. $\mathrm{Sq}^i(\Sigma\alpha) = \Sigma\left(\mathrm{Sq}^i \alpha\right)$ *where $\Sigma : H^n(X;\mathbb{Z}/2) \to H^{n+1}(X;\mathbb{Z}/2)$ is the suspension isomorphism,*
5. $\mathrm{Sq}^i \alpha = \alpha^2$ *if $i = |\alpha|$ and* $\mathrm{Sq}^i \alpha = 0$ *if $i > |\alpha|$,*
6. Sq^1 *is the $\mathbb{Z}/2$ Bockstein homomorphism β associated with the coefficient sequence* $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$.

We omit the proof. See section 4.L of [9].

3.2 Steenrod Squares in Projective Spaces

This results in this section is from [21]. In this section, we examine the behaviour of the Steenrod squares on projective spaces. In the next section, we will show these properties imply that S^{n-1} cannot have $\rho(n)$ vector fields when $d = 0$, thus proving Theorem 3.1.

We know that the cohomology ring $H^*(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$, where $|\alpha| = 1$ and α^j is the generator of $H^j(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2)$.

Theorem 3.13. *If $0 \leq i \leq j$ and $i + j \leq n$, then $Sq^i \alpha^j = \binom{j}{i} \alpha^{i+j} \in \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$.*

Proof. The case $j = 1$ is straightforward: $\mathrm{Sq}^0 \alpha = \alpha$ and $\mathrm{Sq}^1 \alpha = \alpha \smile \alpha$, since $|\alpha| = 1$.

Proceed by induction on j. Using the Cartan formula,

$$\mathrm{Sq}^i \alpha^j = \mathrm{Sq}^i(\alpha \smile \alpha^{j-1}) = \sum_{k=0}^{i} \mathrm{Sq}^k \alpha \smile \mathrm{Sq}^{i-k} \alpha^{j-1} = \mathrm{Sq}^0 \alpha \smile \mathrm{Sq}^i \alpha^{j-1} + \mathrm{Sq}^1 \alpha \smile \mathrm{Sq}^{i-1} \alpha^{j-1},$$

since $\mathrm{Sq}^k u = 0$ if $k > |u|$. Then the induction hypothesis gives us that

$$\begin{aligned}\mathrm{Sq}^0 \alpha \smile \mathrm{Sq}^i \alpha^{j-1} + \mathrm{Sq}^1 \alpha \smile \mathrm{Sq}^{i-1} \alpha^{j-1} &= \binom{1}{0} \alpha \smile \binom{j-1}{i} \alpha^{i+j-1} + \binom{1}{1} \alpha^2 \smile \binom{j-1}{i-1} \alpha^{i+j-2} \\ &= \left(\binom{j-1}{i} + \binom{j-1}{i-1} \right) \alpha^{i+j} = \binom{j}{i} \alpha^{i+j}. \qquad \square\end{aligned}$$

Lemma 3.14. *Let $i = \sum_{k=0}^{m} a_k 2^k$ and $j = \sum_{k=0}^{m} b_k 2^k$ be the binary expansions of i and j. (So a_k and b_k are all 0 or 1.) The binomial coefficient $\binom{j}{i} \equiv 1 \mod 2$ if and only if $a_k = 1$ implies $b_k = 1$ for all k.*

Proof. Note that $\binom{j}{i}$ is the coefficient of u^i in the expansion of $(1 + u)^j$. Calculation mod 2, $(1 + u)^2 = 1 + u^2$ and consequently $(1 + u)^{2^k} = 1 + u^{2^k}$. Therefore,

$$(1 + u)^j = \prod_{k=0}^{m} (1 + u)^{b_k 2^k} = \prod_{k=0}^{m} (1 + b_k u^{2^k}). \tag{3.1}$$

Let $k_1, ..., k_p$ be those values of k for which $a_k = 1$. Then $u^j = u^{2^{k_1}} ... u^{2^{k_p}}$ and so the coefficient of u^j in equation 3.1 is $b_{k_1} ... b_{k_p}$ by the uniqueness of the binary expansion. Thus, $\binom{j}{i} \equiv b_{k_1} ... b_{k_p} \equiv 1 \mod 2$ if and only if $b_{k_1}, ..., b_{k_p}$ are all 1. $\square$

Theorem 3.15. *If $n = (2a + 1)2^b$ with $a > 0$ then in $\mathbb{R}\mathrm{P}^{n-1}$ we have*

$$\mathrm{Sq}^{2^b} \alpha^{n-2^b-1} = \alpha^{n-1} \text{ and } \mathrm{Sq}^j \alpha^{n-j-1} = 0 \text{ for } 0 < j < 2^b.$$

Proof. To prove the first assertion, note that 2^b is a non-zero term of the binary expansion of $n - 1 - 2^b$ since

$$n - 1 - 2^b = 2^{b+1} a - 1 = 2^{b+1} - 1 + 2^{b+1}(a - 1) = \sum_{i=0}^{b} 2^i + 2^{b+1}(a - 1).$$

Then Lemma 3.14 implies $\binom{n-1-2^b}{2^b} \equiv 1 \mod 2$ and Theorem 3.13 gives us that $\mathrm{Sq}^{2^b}\alpha^{n-1-2^b} = \alpha^{n-1}$.

To prove the second statement, suppose $j = 2^s(2t+1)$ where $0 \le s < b$. Then we know the coefficient of 2^s in the binary expansion of j is 1. (Why? If $j = \sum_{i=0}^m a_i 2^i$ is the binary expansion of j, then $a_i = 0$ for all $i < s$. It follows that $a_s = 0$ implies 2^{s+1} divides j. So a_s must be 1.) Since

$$n-1 = 2^b - 1 + 2^{b+1}a = \sum_{i=0}^{b-1} 2^i + 2^{b+1}a,$$

the coefficient of 2^s in the binary expansion $n-1$ is 1. Thus, the coefficient of 2^s in the binary expansion of $n-j-1$ is 0. Lemma 3.14 implies $\binom{n-j-1}{j} \equiv 0 \mod 2$ and Theorem 3.13 gives the required result. $\square$

3.3 Proving the special case

The following is a simplified version of Steenrod and Whitehead's original proof, adapted from [9, page 494].

Proof. When $a = 0$, the result is trivially true. In this case, $n = 2^b$ and $\rho(n) = 2^b$. It is not possible to have 2^b orthogonal vectors tangent to a point in $S^{2^b-1} \subset \mathbb{R}^{2^b}$ – there simply are not enough dimensions. Therefore, we can assume $a > 0$. Let $k = \rho(n) + 1$.

Suppose we have $k-1$ vector fields on S^{n-1}. By Lemma 1.40, equivalently suppose we have section $f : S^{n-1} \to V_{n,k}$ of the bundle $p : V_{n,k} \to S^{n-1}$. Then

$$H^{n-1}(S^{n-1};\mathbb{Z}_2) \xrightarrow{p^*} H^{n-1}(V_{n,k};\mathbb{Z}_2) \xrightarrow{f^*} H^{n-1}(S^{n-1};\mathbb{Z}_2)$$

is the identity. It follows that f^* is surjective. We can always deform f to be a cellular map and then its image will be contained in the $(n-1)$-skeleton of $V_{n,k}$.

We know that $2k - 1 = 2\rho(n) + 1 = 2^{b+1} + 1 \le n$. So Proposition 1.41 tells us that the $(n-1)$-skeleton of $V_{n,k}$ is $\mathbb{R}\mathrm{P}^{n-1}_{n-k}$. Thus, by deforming f if necessary, we obtain a map $g : S^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}_{n-k}$ with

$$S^{n-1} \xrightarrow{g} \mathbb{R}\mathrm{P}^{n-1}_{n-k} \xrightarrow{p|_{\mathbb{R}\mathrm{P}^{n-1}_{n-k}}} S^{n-1}$$

the identity. Then g^* is surjective on $H^{n-1}(-;\mathbb{Z}_2)$. Since

$$H^{n-1}(\mathbb{R}\mathrm{P}^{n-1}_{n-k};\mathbb{Z}_2) \cong H^{n-1}(S^{n-1};\mathbb{Z}_2) \cong \mathbb{Z}_2,$$

it follows that g^* is an isomorphism on H^{n-1}.

Now, the inclusion $i : \mathbb{R}\mathrm{P}^{n-1}_{n-k} \to \mathbb{R}\mathrm{P}^{n-1}$ induces an isomorphism on $H^{n-1}(-;\mathbb{Z}_2)$. Thus, $g^*i^* : H^{n-1}(\mathbb{R}\mathrm{P}^{n-1};\mathbb{Z}_2) \xrightarrow{\cong} H^{n-1}(S^{n-1};\mathbb{Z}_2)$ is an isomorphism.

By Theorem 3.15, we know that

$$\mathrm{Sq}^{k-1} : H^{n-k}(\mathbb{R}\mathrm{P}^{n-1};\mathbb{Z}_2) \to H^{n-1}(\mathbb{R}\mathrm{P}^{n-1};\mathbb{Z}_2)$$

is non-zero. But

$$\mathrm{Sq}^{k-1} : H^{n-k}(S^{n-1};\mathbb{Z}_2) \to H^{n-1}(S^{n-1};\mathbb{Z}_2)$$

is obviously zero. We then have a contradiction by using the naturality of Steenrod squares. Specifically, we know that $g^* i^* \mathrm{Sq}^{k-1} \alpha^{n-k} = g^* i^* \alpha^{n-1} \neq 0$, where α^{n-k} is the generator of $H^{n-k}(\mathbb{R}\mathrm{P}^{n-1};\mathbb{Z}_2)$. But $\mathrm{Sq}^{k-1} g^* i^* \alpha^{n-k}$ must be 0. $\square$

If we replaced $k = 2^c + 1$ with $j \leq 2^c$, then

$$\begin{gathered} Sq^{j-1} : H^{n-j}(\mathbb{R}\mathrm{P}^{n-1};\mathbb{Z}_2) \to H^{n-1}(\mathbb{R}\mathrm{P}^{n-1};\mathbb{Z}_2), \\ \alpha^{n-j} \mapsto 0, \end{gathered}$$

by Theorem 3.15 and so we would no longer gain a contradiction. This shows that the upper bound of 2^c on the number of vector fields is the best possible provided by this argument. So this argument can only prove that $\rho(n)$ is the upper bound when $d = 0$. We need more sophisticated tools (i.e. K-theory) to prove the general case.

Chapter 4

The Stable Homotopy Category

We are now at the point where we need to develop some significant amount of theory before we can make more progress on our vector fields problem. The next three chapters—on the stable homotopy category, duality and spectral sequences—may seem to be a detour, but they have important applications to our task, as well as many other areas of algebraic topology.

In this chapter, we will introduce the stable homotopy category. There are a number of constructions of the stable homotopy category. All these constructions first build a category of objects, called spectra. Then they replace the morphisms in this category with some notion of homotopy classes of maps. The resulting category is called the stable homotopy category **Ho**(**Spectra**). These constructions are analogous to defining the category **CW** of CW complexes and then using this to construct the category **Ho**(**CW**) of CW complexes with homotopy classes of maps.

All of these constructions result in equivalent categories. So we can talk about *the* stable homotopy category.

We will primarily present the historical construction, given first by Boardman [23][†]. While the stable homotopy category enjoys a symmetric smash product, Boardman's category of spectra does not. It was only in 1997 when a category of spectra with a symmetric smash product was discovered by Elmendorf, Kriz, Mandell and May [7]. We will briefly present the category of symmetric spectra, first given in [12], to highlight the advantages of these modern spectra.

As indicated by the name, the stable homotopy category is the natural environment for stable homotopy theory. One advantage is that the suspension functor

$$\Sigma : \mathbf{Ho}(\mathbf{Spectra}) \to \mathbf{Ho}(\mathbf{Spectra})$$

is an equivalence, as we shall prove later. So we can invert suspensions and consequently many phenomenon are stable under suspension in **Ho**(**Spectra**). Suspension on topological spaces is not so well behaved. In this sense, **Ho**(**Spectra**) is the 'stabilisation' of the classical homotopy category **Ho**(**Top**).

[†]However, Adams attributes the notion of spectra to Lima (in [17]) and G. W. Whitehead [2, p. 131]

In this chapter, we assume every space is locally compact Hausdorff.

4.1 Boardman's spectra

4.1.1 Definitions

Definition 4.1. A *(Boardman) spectrum* E is a sequence $\{E_n\}$ of based spaces (called components) with structure maps

$$\sigma_n : \Sigma E_n \to E_{n+1}.$$

The index n may vary over $\mathbb{Z}$ or $\mathbb{N}$. We will show later that the choice does not matter. Usually, we will index by $\mathbb{N}$.

Definition 4.2. For a based space (X, x_0), the *loopspace* ΩX is the space of loops in X based at x_0 (that is, the space of based maps $(S^1, s_0) \to (X, x_0)$), with the compact-open topology. The constant loop is taken as the basepoint of ΩX.

Define $\Omega_0 X$ to be the connected component of ΩX containing the basepoint.

By loopspace-suspension adjunction, we can write the structure maps in the form

$$\sigma'_n : E_n \to \Omega E_{n+1}.$$

Definition 4.3. A spectrum E is an *Ω-spectrum* (or *omega-spectrum*) if $\sigma'_n : E_n \to \Omega E_{n+1}$ is a weak equivalence for all n. If, in addition, all the components are connected, then σ'_n maps E_n into $\Omega_0 E_{n+1}$ and E is an *Ω_0-spectrum.*

Example 4.4. *Given a space X, we can form a spectrum $\Sigma^\infty X$ with components*

$$(\Sigma^\infty X)_n := \Sigma^n X$$

for $n \in \mathbb{N}$. The structure maps are simply the identity. We call $\Sigma^\infty X$ the suspension spectrum of X. In particular, $\Sigma^\infty S^0$ is the sphere spectrum and $\Sigma^\infty *$ is the spectrum where every component is a point $*$.

From here on, we will often denote $\Sigma^\infty X$ by X as well, when the meaning is clear from the context.

We define the wedge sum of spectra componentwise:

Definition 4.5. The *wedge sum* of spectra E and E' is the spectrum $E \vee E'$ with spaces

$$(E \vee E')_n := E_n \vee E'_n.$$

Since $\Sigma(X \vee Y) \cong \Sigma X \vee \Sigma Y$, for spaces X and Y, the structure maps are simply

$$\sigma^E_n \vee \sigma^{E'}_n.$$

Definition 4.6. A *map of spectra* $f : E \to E'$, of degree 0, is a sequence of maps $f_n : E_n \to E'_n$ making

$$\begin{tikzcd} \Sigma E_n \arrow[r] \arrow[d, "\Sigma f_n"'] & E_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma E'_n \arrow[r] & E'_{n+1} \end{tikzcd}$$

commute.

A *map of spectra* $f : E \to E'$ of degree r is a sequence of maps $f_n : E_n \to E'_{n-r}$ making the analogous diagram commute.

This is not very well behaved. We demonstrate this with the following example. Define the sphere spectrum S with $S_n = S^n$ and $\sigma_n = \mathsf{id}_{S^{n+1}}$.

Define a modified version S' of the sphere spectrum with

$$S'_n = \begin{cases} S^n & \text{if } n \geq n_0, \\ * & \text{if } n < n_0, \end{cases}$$

for some constant n_0. The structure maps are trivial

$$\sigma_n = \begin{cases} * \xrightarrow{\mathsf{id}} * & \text{if } n < n_0 - 1, \\ * \xrightarrow{\mathsf{tr}} S^{n_0+1} & \text{if } n = n_0 - 1, \\ S^n \to S^n & \text{if } n \geq n_0. \end{cases}$$

We have an obvious map $f : S' \to S$. For $n < n_0$, $f_n : * \to S^n$ is the trivial map and $f_n : S^n \to S^n$ is the identity for $n \geq n_0$. This is indeed a map of spectra since

$$\begin{tikzcd} \Sigma S'_{n_0-1} = * \arrow[r] \arrow[d, "f_{n_0-1}"'] & S'_{n_0} = S^{n_0} \arrow[d, "\mathsf{id}"] \\ \Sigma S_{n_0-1} = S^{n_0} \arrow[r] & S^{n_0} \end{tikzcd}$$

commutes.

But if we want a map $g : S \to S'$, we need that

$$\begin{tikzcd} S^{n_0} = \Sigma S_{n_0-1} \arrow[r, "\mathsf{id}"] \arrow[d, "\Sigma g_{n_0-1}"'] & S_{n_0} = S^{n_0} \arrow[d, "g_{n_0}"] \\ \Sigma S'_{n_0-1} = * \arrow[r, "\mathsf{tr}"] & S'_{n_0} = S^{n_0} \end{tikzcd}$$

commutes. This implies that g_{n_0} must be trivial. By induction g_n must be trivial for all $n \geq n_0$.

So we have a good map $f : S' \to S$ which induces an isomorphism on homotopy groups. (Later, we will define the homotopy groups of a spectrum E as $\pi_r(E) := \operatorname{colim}_{n\to\infty} \pi_{r+n} E_n$. In particular, the homotopy groups of S and S' are just the stable homotopy groups of spheres.) We would then like f to have some sort of inverse, since in the category of CW complexes, we have this property†. But the only map $S \to S'$ is the trivial map, as we have seen.

4.1.2 Constructing the Stable Homotopy Category

So if we want the induced stable homotopy category to behave well, it will take a bit of work to develop a good definition of homotopy classes of maps. First, we restrict our attention to CW spectra.

Definition 4.7. A spectrum E is a *CW spectrum* if each component E_n is a based CW-complex and each structure map $\sigma_n : \Sigma E_n \to E_{n+1}$ is a homeomorphism from ΣE_n to a subcomplex of E_{n+1}.

Unsurprisingly, the connection between CW spectra and CW spaces is deep. Throughout this section, we will see that many of the nice properties of CW spaces carry over to spectra.

Definition 4.8. A *subspectrum* A of a CW spectrum E is a spectrum with components $A_n \subset E_n$ a subcomplex.

A subspectrum A is *cofinal* in E if for each n and each finite subcomplex $K \subset E_n$, there is some m such that the map

$$\Sigma^m E_n \xrightarrow{\Sigma^{m-1}\epsilon_n} \Sigma^{m-1} E_{n+1} \xrightarrow{\Sigma^{m-2}\epsilon_{n+1}} \dots \xrightarrow{\epsilon_{n+m-1}E_{n+m}} E_{n+m}$$

sends $\Sigma^m K$ into A_{n+m}.

Intuitively, A is cofinal in E if A eventually contains every subcomplex $K \subset E_n$.

Definition 4.9. Let E be a CW spectrum and E' any spectrum. Suppose E_1 and E_2 are cofinal subspectra of E. Two maps $f_1 : E_1 \to E'$, $f_2 : E_2 \to E'$ of the same degree are *equivalent* if there exists a cofinal subspectrum E_3 contained in both E_1 and E_2 such that the restrictions of f_1 and f_2 to E_3 are equal.

The previous definition gives an equivalence relation. The proof of this is straightforward, using the fact that the intersection of two cofinal subspectra is a cofinal subspectrum.

Definition 4.10. An *emap* from a CW spectrum E to a spectrum E' is an equivalence class of maps from cofinal subspectra of E to E'.

Note the equivalence class consist of maps of the same degree r, by definition. So we can define the degree of the emap to be r.

All of this work amounts to saying that, when defining an emap f, if you have a cell c in E_n, we need not define f on c at once; you can wait till E_{m+n} before defining the map on $\Sigma^m c$. The slogan from [2] is, "cells now—maps later."

†This is Whitehead's theorem, which states that a weak equivalence between CW complexes is a homotopy equivalence.

Proposition 4.11. *Composition of maps of spectra determines a well-defined composition of emaps.*

That is, it is possible to define composition of emaps as $[f] \circ [g] = [f \circ g]$, where f, g are representatives of the emaps $[f]$ and $[g]$ respectively. We omit the proof; see [25, Proposition 1.5, §3].

We want to define homotopy of emaps. Recall that, in **Top**, a homotopy of maps $f, g : X \to Y$ is a map $h : I \times X \to Y$ satisfying $h(0, -) = f$ and $h(1, -) = g$. We will define an analogous concept for spectra.

Let $fl : X \wedge Y \to Y \wedge X$ be the flip map. Let the cylinder spectrum $\mathrm{Cyl}(E)$ of E be the spectrum with components $I_+ \wedge E_n$ and structure maps

$$S^1 \wedge I_+ \wedge E_n \xrightarrow{fl \wedge \mathsf{id}_{E_n}} I_+ \wedge S^1 \wedge E_n \xrightarrow{\mathsf{id}_{I_+} \wedge \sigma_n} I_+ \wedge E_{n+1},$$

Given $f : E \to E'$, define $\mathrm{Cyl}(f) : \mathrm{Cyl}(E) \to \mathrm{Cyl}(E')$ in the obvious way.

Proposition 4.12. *There are injections (of degree 0)*

$$i_0, i_1 : E \to \mathrm{Cyl}(E)$$

corresponding to the two ends of the cylinder.

Proof. We define i_0 by

$$\begin{aligned} i_{0,n} : E_n &\to \mathrm{Cyl}(E)_n \\ x &\mapsto (0, x). \end{aligned}$$

We need to show that i_0 is a map of spectra.

Since the smash product is symmetric, we can consider σ_n as a map $E_n \wedge S^1 \to E_{n+1}$. Then we need to verify that the square

$$\begin{tikzcd} E_n \wedge S^1 \arrow[r, "\sigma_n"] \arrow[d, "i_{0,n} \wedge \mathsf{id}_{S^1}"'] & E_{n+1} \arrow[d, "i_{0,n+1}"] \\ I_+ \wedge E_n \wedge S^1 \arrow[r, "\mathsf{id}_{[0,1]_+} \wedge \sigma_n"] & I_+ \wedge E_{n+1} \end{tikzcd}$$

commutes. It is easy to verify this element-wise:

$$\begin{tikzcd} (x, s) \arrow[r, mapsto, "\sigma_n"] \arrow[d, mapsto, "i_{0,n} \wedge \mathsf{id}_{S^1}"'] & \sigma_n(x, s) \arrow[d, mapsto, "i_{0,n+1}"] \\ (0, x, s) \arrow[r, mapsto, "\mathsf{id}_{[0,1]_+} \wedge \sigma_n"] & (0, \sigma_n(x, y)). \end{tikzcd}$$

The i_1 case is analogous. □

It is straightforward to check that the cylinder spectrum $\mathrm{Cyl}(E)$ of a CW-spectrum E is also a CW-spectrum. Thus, we can consider the emaps from $\mathrm{Cyl}(E)$.

Definition 4.13. Two emaps $f, g : E \to E'$ of the same degree are *homotopic* if there exists a emap $h : \mathrm{Cyl}(E) \to F$ such that $f = h \circ i_0$ and $g = h \circ i_1$.

The standard proof that homotopy defines an equivalence relation extends to the case of emaps. The equivalence classes of this relation are called homotopy classes. The set of homotopy classes of emaps $E \to F$ of degree 0 is denoted by $[E, F]$ and the set of homotopy classes of emaps of degree r is denoted by $[E, F]_r$.

Proposition 4.14. *Composition of emaps determines a well-defined composition of homotopy classes.*

Again, this means we can define the composition of homotopy classes $[f]$ and $[g]$ as $[g \circ f]$. The standard proof of this property also extends to the case of emaps.

Definition 4.15. The *stable homotopy category* has CW spectra as objects and homotopy classes of emaps of degree 0 as morphisms.

We can define another version of the stable homotopy category, where we do not force the degree of the emaps to be 0. Then the homsets are $\mathbb{Z}$-graded, by their degree. If you compose a morphism of degree r and a morphism of degree s then you have a morphism of degree $r + s$.

We see that we have now solved our motivating problem: that there was no good inverse to the weak equivalence $f : S' \to S$. Now we know that S' is a cofinal subspectrum of S, so there is a map $S \to S'$ in the stable homotopy category corresponding to the identity map $S' \to S'$ of spectra. This is the inverse of f in the stable homotopy category. The following proposition makes this argument rigorous.

Proposition 4.16. *If E' is cofinal in E then E' and E are isomorphic in the stable homotopy category.*

Given the working above, this result should now be intuitive. See [2, §, part III] for a proof.

Example 4.17. *Given a CW-spectrum E, define E' by*

$$E'_n = \begin{cases} E_n & \text{if } n \geq 0, \\ * & \text{if } n < 0. \end{cases}$$

We know E' is cofinal in E and therefore isomorphic in the stable homotopy category. So it does not matter whether we index spectra by $\mathbb{Z}$ or $\mathbb{N}$.

4.1.3 Homotopy Groups of Spectra

Definition 4.18. Define the *r-th homotopy group* of a spectrum E as

$$\pi_r(E) = \operatorname*{colim}_{n\to\infty} \pi_{r+n}(E_n).$$

On the right side, the direct limit is taken over the maps

$$\begin{aligned}[S^{r+n}, E_n] &\xrightarrow{\sigma_E \circ \Sigma -} [S^{r+n+1}, E_{n+1}],\\ f &\longmapsto \sigma_E \circ \Sigma f.\end{aligned}$$

Since the direct limit of a sequence of abelian groups always exists (Proposition A.4), the homotopy groups are always well defined. However, the sequence may not stabilise as the following example illustrates.

Example 4.19. *Let E be the spectrum with components $E_n = \vee_n S^n$ and structure maps $\sigma : \Sigma E_n \to E_{n+1}$ inclusion of the first n wedge summands. Then*

$$\pi_0(E) = \operatorname*{colim}_{n\to\infty} \pi_n(E_n) = \operatorname*{colim}_{n\to\infty} \oplus_n \mathbb{Z} = \oplus_{n\in\mathbb{N}} \mathbb{Z}.$$

Proposition 4.20 [2, Proposition 2.8, part III]**.** *Suppose X is a finite CW complex and E a spectrum. Then there is a natural 1-to-1 correspondence*

$$[\Sigma^\infty X, E]_r \leftrightarrow \operatorname*{colim}_{n\to\infty} [\Sigma^{n+r} X, E_n].$$

Analogous to the above definition, on the right side, the direct limit is taken over the maps

$$[\Sigma^{n+r} X, E_n] \xrightarrow{\sigma_E \circ \Sigma -} [\Sigma^{n+r+1} X, E_{n+1}].$$

For a proof, see [2, Proposition 2.8, part III]. An easy corollary is that $\pi_r(E) = [S, E]_r$, where S is the sphere spectrum.

A map of spectra $f : E \to E'$ induces a homomorphism between homotopy groups $f_* : \pi_r(E) \to \pi_r(E')$. There are maps $\pi_{n+r} E_n \to \pi_{n+r} E'_n$ induced by the component f_n. Then taking colimits gives an induced map

$$f^* : \operatorname*{colim}_{n\to\infty} \pi_{n+r} E_n \to \operatorname*{colim}_{n\to\infty} \pi_{n+r} F_n$$

between the homotopy groups of spectra.

4.1.4 Brown's Representability Theorem

Every spectrum E represents a reduced cohomology theory $\widetilde{E}^*$, called the E cohomology, defined by

$$\widetilde{E}^r(X) = \operatorname*{colim}_{n\to\infty} [\Sigma^n X, E_{n+r}].$$

(The direct limit is taken over the same maps as in Proposition 4.20.) If E is an Ω-spectrum, then this simplifies to

$$\widetilde{E}^r(X) = [X, E_r],$$

where the right side is a homotopy classes of maps of spaces, since the structure maps σ_E are weak equivalences.

The corresponding unreduced cohomology theory is given by

$$E^r(X) = [X_+, E_r].$$

Brown's representability theorem states that we can go in the other direction too and build Ω-spectra from reduced cohomology theories.

Theorem 4.21 [9, Theorem 4E.1]**.** *Let $\widetilde{E}^*$ be a reduced cohomology theory. Then there exist an Ω-spectrum E and natural isomorphism $\widetilde{E}^r X \cong [X, E_r]$. Furthermore, the spaces of E are unique up to homotopy equivalence.*

Therefore, this construction is an equivalence (in the category theory sense) between Ω-spectra and reduced cohomology theories.

Examples

The sphere spectrum S represents

$$E^r(X) = \lim_k [\Sigma^r X, S^{r+k}]$$

the r-th stable cohomotopy group.

Recall that the loopspace $\Omega K(\pi, n+1)$ is an Eilenberg-MacLane space of type $K(\pi, n)$. This allows us to make the following definition.

Definition 4.22. Given a group π, we define the *Eilenberg-MacLane spectrum $H\pi$* to be the omega spectrum whose n-th space is $K(\pi, n)$ and whose structure maps are

$$\sigma'_n : K(\pi, n) \to \Omega K(\pi, n+1).$$

Ordinary cohomology is represented by the Eilenberg-MacLane spectrum:

$$\widetilde{H}^r(X; G) \cong [X, K(G, r)]$$

Spectra Represent Homology (as well)

A spectrum E also represents a reduced homology theory defined by

$$E_r(X) = \pi_r(E \wedge X).^\dagger$$

Proposition 4.23. *The Eilenberg-MacLane spectra $H\mathbb{Z}$ represents ordinary homology with coefficients in $\mathbb{Z}$*

$$\pi_r(H\mathbb{Z} \wedge X) \cong \widetilde{H}_r(X, \mathbb{Z}).$$

For a proof see [25, §11].

---

†As before, the smash product $\wedge$ takes a spectrum E and a based space X, and gives a spectrum defined by $(E \wedge X)_n = E_n \wedge X$ with structure maps $S^1 \wedge E_n \wedge X \xrightarrow{\sigma_n \wedge \mathrm{id}_X} E_{n+1} \wedge X$.

4.1.5 Weak Equivalence of Spectra

Definition 4.24. A map $f : E \to F$ of spectra is a *weak equivalence* if it induces isomorphisms on all homotopy groups.

Every space has a CW approximation [9, page 352]. This result carries over to spectra as well:

Theorem 4.25. *Every spectrum is weakly equivalent to a CW-spectrum.*

So we are not losing anything by our restriction to CW spectrum. We can consider every spectra as an object of the stable homotopy category, by passing to a weakly equivalent CW spectra.

In [2, part III], Adams defines weak equivalence of spectra as follows.

Definition 4.26. A map $f : E \to F$ between spectra E, F is a *weak equivalence* if the induced map

$$\begin{aligned} f_* : [X, E]_* &\to [X, F]_* \\ g &\mapsto f \circ g \end{aligned}$$

is a bijection, for all CW-spectra X.

This is equivalent to our definition, by the following theorem:

Theorem 4.27 [2, Theorem 3.4, part III]**.** *Let $f : E \to F$ be a function between spectra such that the induced map $f_* : \pi_*(E) \to \pi_*(F)$ is an isomorphism. Then for any CW-spectrum X,*

$$f_* : [X, E]_* \to [X, F]_*$$

is a bijection.

By a standard category theory fact†, this implies that weak equivalences are isomorphisms in the stable homotopy category. This proves the following corollary:

Corollary 4.28. *Let $f : E \to F$ be a map such that $f_n : E_n \to F_n$ is a homotopy equivalence for each n. Then f is an isomorphism in the stable homotopy category.*

Moreover, all isomorphisms in the stable homotopy category are weak equivalences†.

4.1.6 Inverse Suspension

One of the advantages of working with spectra, instead of spaces, is that we can invert suspensions.

†The category theory fact is 'if $f_* : \mathscr{C}(X, E) \xrightarrow{f\circ -} \mathscr{C}(X, F)$ is a bijection, then f is an isomorphism'. Proof: define f^{-1} as the map in $\mathscr{C}(F, E)$ sent to $\mathsf{id} \in \mathscr{C}(F, F)$ by f_*.

†If $f : E \to F$ is an isomorphisms in the stable homotopy category, then there is $f^{-1} : F \to E$ such that $f \circ f^{-1} = \mathsf{id}_F$ and $f^{-1} \circ f = \mathsf{id}_E$. Then the induced maps on homotopy groups satisfy $f_* \circ f_*^{-1} = \mathsf{id}$ and $f_*^{-1} \circ f_* = \mathsf{id}$. So f_* is an isomorphism of homotopy groups.

Definition 4.29. Let E be a spectrum with components E_n and structure maps σ_n. The *suspension* ΣE of E has components

$$(\Sigma E)_n := S^1 \wedge E_n$$

and structure maps

$$S^1 \wedge S^1 \wedge E_n \xrightarrow{fl \wedge \mathrm{id}_{E_n}} S^1 \wedge S^1 \wedge E_n \xrightarrow{\mathrm{id}_{S^1} \wedge \sigma_n} S^1 \wedge E_{n+1}$$

where $fl : S^1 \wedge S^1 \to S^1 \wedge S^1$ is the flip map.

The *desuspension* $\Sigma^{-1}E$ of E has components $(\Sigma^{-1}E)_n = E_{n-1}$ and n-th structure map given by the $(n-1)$-st structure map σ_{n-1} of E.

Proposition 4.30. *There are natural isomorphisms in the stable homotopy category*

$$E \xrightarrow{\cong} \Sigma\Sigma^{-1}E,$$
$$\Sigma^{-1}\Sigma E \xrightarrow{\cong} E.$$

We define Σ and Σ^{-1} on maps as well, in the obvious way. Given a map of spectra $f : E \to E'$ of degree r with components $f_n : E_n \to E'_n$, the suspension $\Sigma f : \Sigma E \to \Sigma E'$ is a map of spectra of degree r with components Σf_n. Similarly, the desuspension $\Sigma^{-1}f : \Sigma^{-1}E \to \Sigma^{-1}E'$ has components f_{n-1}. It is not hard to see that this gives well defined functors Σ and Σ^{-1} in the stable homotopy category.

Corollary 4.31. *Suspension induces a bijection*

$$\Sigma : [E, E']_* \xrightarrow{\cong} [\Sigma E, \Sigma E']_*$$
$$[f] \mapsto [\Sigma f].$$

So we have an adjunction $[\Sigma E, E'] \cong [E, \Sigma^{-1}E']$. The analogous statement for spaces is the suspension-loopspace adjunction $[\Sigma X, Y] \cong [X, \Omega Y]$. So we can think of $\Sigma^{-1}E'$ as the loopspace of the spectrum E'. If E' is an Ω spectrum, then in fact $(\Sigma^{-1}E')_n = E'_{n-1}$ is weakly equivalent to $\Omega E'_n$. So there is an isomorphism in the stable homotopy category between $\Sigma^{-1}E'$ and the spectrum formed by taking loopspaces of each component of E'.

Proof. The isomorphisms of the Proposition 4.30 give a bijection

$$[\Sigma^{-1}\Sigma E, \Sigma^{-1}\Sigma E']_* \xrightarrow{\cong} [E, E']_*.$$

Using this, we can consider

$$\Sigma^{-1} : [\Sigma E, \Sigma E']_* \to [\Sigma^{-1}\Sigma E, \Sigma^{-1}\Sigma E']_* \cong [E, E'].$$

This map is the inverse of the suspension Σ, so we are done. ∎

We will prove Proposition 4.30 by introducing a 'fake' suspension Σ_f: given a spectrum E, the fake suspension $\Sigma_f E$ has components

$$(\Sigma E)_n := S^1 \wedge E_n$$

and structure maps

$$S^1 \wedge S^1 \wedge E_n \xrightarrow{\mathsf{id}_{S^1} \wedge \sigma_n} S^1 \wedge E_{n+1}.$$

So ΣE and $\Sigma_f E$ are identical, except the structure maps of ΣE first flip $S^1 \wedge S^1$. This flip is critical if we want to define a compatible smash product on spectra.

Proposition 4.32. *There is a natural isomorphism between $\Sigma_f E$ and ΣE in the stable homotopy category.*

We will not prove this result. See [25, Proposition 1.4, §4]. The proof of Proposition 4.30 follows from the following lemma:

Lemma 4.33. *Proposition 4.30 holds when Σ is replaced by Σ_f.*

Proof. To begin, note that the n-th space of $\Sigma_f \Sigma^{-1} E$ and $\Sigma^{-1}\Sigma_f E$ are ΣE_{n-1}. Moreover, the n-th structure maps of $\Sigma_f \Sigma^{-1} E$ and $\Sigma^{-1}\Sigma_f E$ are both $\mathsf{id}_{S^1} \wedge \sigma_{n-1}$. Therefore, we obtain a canonical isomorphism

$$\Sigma_f \Sigma^{-1} E \xrightarrow{\cong} \Sigma^{-1}\Sigma_f E.$$

Using this result, it suffices to prove $E \xrightarrow{\cong} \Sigma_f \Sigma^{-1} E$.

Note that the n-th space of $\Sigma_f \Sigma^{-1} E$ is $S^1 \wedge E_{n-1}$ and the n-th structure map is $\mathsf{id}_{S^1} \wedge \sigma_{n-1}$.

We will construct a weak equivalence $f : \Sigma^{-1}\Sigma_f E \to E$. Then by previous results, f will be an isomorphism in the stable homotopy category.

Define the n-th component of f to be the structure map $\sigma_{n-1} : \Sigma E_{n-1} \to E_n$. This is a map of spectra since the relevant square

$$\begin{tikzcd}
S^1 \wedge S^1 \wedge E_{n-1} \arrow[r, "\mathsf{id}_{S^1} \wedge \sigma_{n-1}"] \arrow[d, "\mathsf{id}_{S^1} \wedge \sigma_{n-1}"'] & S^1 \wedge E_n \arrow[d, "\sigma_n"] \\
S^1 \wedge E_n \arrow[r, "\sigma_n"] & E_{n+1}.
\end{tikzcd}$$

obviously commutes.

We will define an inverse $\phi_r : \pi_r E \to \pi_r \Sigma^{-1}\Sigma_f E$ of the induced map $f_* : \pi_r \Sigma^{-1}\Sigma_f E \to \pi_r E$. Suspension induces a map $\Sigma_* : \pi_{n-1+r} E_{n-1} \to \pi_{n+r} \Sigma E_{n-1}$. Taking colimits as $n \to \infty$, we get our map

$$\phi_r : \pi_r E = \operatorname*{colim}_{n\to\infty} \pi_{n-1+r} E_{n-1} \to \operatorname*{colim}_{n\to\infty} \pi_{n+r} \Sigma E_{n-1} = \pi_r \Sigma^{-1}\Sigma_f E.$$

Then f_* and ϕ_r are inverses by the category theoretic fact that given a diagram

$$\begin{tikzcd}
\cdots \arrow[rr] & & A_n \arrow[rr] \arrow[dr] & & A_{n+1} \arrow[r] \arrow[dr] & \cdots \\
\cdots \arrow[r] & B_n \arrow[ur] \arrow[rr] & & B_{n+1} \arrow[ur] \arrow[rr] & & \cdots
\end{tikzcd}$$

the diagonal arrows induce inverse isomorphisms on the colimits

$$\operatorname*{colim}_{n\to\infty} A_n \cong \operatorname*{colim}_{n\to\infty} B_n.$$

This fact follows straightforwardly from the universal properties of colimits.

In our case, we have constructed the diagram

$$\begin{tikzcd}
\cdots \arrow[rr] & & \pi_{n-1+r}E_{n-1} \arrow[rr] \arrow[dr, "\Sigma_*"] & & \pi_{n+r}E_n \arrow[r] \arrow[dr] & \cdots \\
\cdots \arrow[r] & \pi_{n-1+r}\Sigma E_{n-1} \arrow[ur, "(f_{n-1})_*"] \arrow[rr] & & \pi_{n+r}\Sigma E_n \arrow[ur, "(f_n)_*"] \arrow[rr] & & \cdots
\end{tikzcd}$$

and so we get inverse isomorphisms

$$\mathrm{colim}_{n\to\infty}\, \pi_{n+r}\Sigma E_{n-1} \underset{\phi_r}{\overset{f_*}{\rightleftarrows}} \mathrm{colim}_{n\to\infty}\, \pi_{n-1+r}E_{n-1}.$$ □

This proves that Σ and Σ^{-1} form an equivalence on the stable homotopy category. Since Σ^{-1} is a shift down of indices, $[\Sigma^{-r}E, E']_* \cong [E, E']_{*-r}$. By the equivalence just proven, we must also have $[\Sigma^r E, E']_* \cong [E, E']_{*+r}$.

The next result will prove useful in later chapters.

Proposition 4.34. *If V and W are vector bundles which determine the same class in $K_{\mathbb{F}}(X)$, then $\mathrm{Th}(V) \cong \mathrm{Th}(W)$, in the stable homotopy category.*

Proof. We only consider the case for real vector bundles, since this will suffice for our needs. The complex case follows analogously. If V and W determine the same class, then $V \oplus \epsilon^n = W \oplus \epsilon^m$, for some trivial bundles ϵ^n, ϵ^m. But we know that,

$$\mathrm{Th}(V \oplus \epsilon^n) \cong \Sigma^n \mathrm{Th}(V).$$

in spaces (by Proposition 1.27). Thus, $\mathrm{Th}(V) \cong \Sigma^{m-n}\mathrm{Th}(W) \cong \mathrm{Th}(W)$, in the stable homotopy category. □

One advantage of working with spectra, as opposed to working with spaces, is that $[E, F]$ always forms an abelian group. Since every CW spectrum E is equivalent to its suspension ΣE and hence also its double suspension $\Sigma^2 E$, we can define an abelian sum operation just as in ordinary homotopy theory.

4.1.7 Cells of CW-Spectra

We outline the concept of stable cells given in [2, part III]. This will be important in understanding the Spanier-Whitehead duals of CW complexes, in the next chapter.

Definition 4.35. Let E be a CW-spectrum and let C_n be the set of cells in E_n other than the base-point. Define a map $C_n \to C_{n+1}$ by $c \mapsto \sigma_n(\Sigma c)$. This is an injection (by definition of CW-spectra), so the direct limit

$$C = \operatorname*{colim}_{n \to \infty} C_n$$

exists by proposition A.5. An element of C is called a *stable cell* of E.

By definition, a stable cell is an equivalence class of cells. Each equivalence class contains at most one cell in E_n. Moreover, if c and c' are cells in E_n and E_m respectively, with $n \leq m$, then c and c' are equivalent if and only if

$$C_n \to C_{n+1} \to \ldots \to C_m$$

maps c to c'.

Definition 4.36. Let $[c]$ be a stable cell represented by $c \in E_n$. Suppose the dimension of c is m. Then the *stable dimension* of $[c]$ is $m - n$.

The stable dimension is well defined by the reasoning in the previous paragraph. This allows us to have cells of negative dimensions in CW spectra.

Proposition 4.37. *A subspectrum $E' \subset E$ is cofinal if and only if $C' \to C$ is a bijection.*

So a cofinal subspectrum has the same stable cells as its superspectrum. This explains why we cared about maps out of cofinal subspectrum, instead of maps out of the superspectrum. We only cared about maps defined on the stable cells. The proof of this proposition is straightforward.

Definition 4.38. A CW spectrum is *finite* if it has are a finite number of stable cells.

4.2 The Smash Product and Symmetric Spectra

We want a smash product that provides a symmetric monoidal structure on the category of spectra. Moreover, a ring spectrum E—a spectrum representing a multiplicative cohomology theory—needs to come with a map $E \wedge E \to E$ which encodes the multiplication in E-cohomology.

More explicitly, given spectra $X = \{X_n\}$ and $Y = \{Y_n\}$, we want the following properties:

1. $X \wedge Y \cong Y \wedge X$,

2. $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$,
3. the sphere spectrum S is the unit: $X \wedge S \cong X \cong S \wedge X$,
4. and these isomorphisms act nicely (i.e. they satisfy the pentagon axioms).

One obvious possible definition is

$$(X \wedge Y)_{2n} := X_n \wedge Y_n \text{ and } (X \wedge Y)_{2n+1} := X_{n+1} \wedge Y_n,$$

with structure maps

$$\sigma_{2n} : \Sigma(X_n \wedge Y_n) = \Sigma X_n \wedge Y_n \xrightarrow{\sigma_{n,X} \wedge \mathsf{id}_{Y_n}} X_{n+1} \wedge Y_n$$
$$\sigma_{2n+1} : \Sigma(X_{n+1} \wedge Y_n) = X_{n+1} \wedge \Sigma Y_n \xrightarrow{\mathsf{id}_{X_{n+1}} \wedge \sigma_{n,Y}} X_{n+1} \wedge Y_{n+1} = (X \wedge Y)_{2(n+1)}.$$

But this definition is not symmetric, we don't have $X \wedge Y \cong Y \wedge X$.

Another possible definition is

$$(X \wedge Y)_n = \bigvee_{i+j=n} X_i \wedge Y_j,$$

with structure maps are

$$\sigma_n : \Sigma(X \wedge Y)_n = \bigvee_{i+j=n} \Sigma(X_i \wedge Y_j) \xrightarrow{\vee(\sigma_X \wedge \sigma_Y)} (X \wedge Y)_{n+1}.$$

This satisfies properties 1. and 2. But it doesn't satisfy property 3. We run into problems with the homotopy groups of $X \wedge S$ and X, which must be the same for property 3 to hold.

So it is no easy task to define a nice smash product on Boardman's category of spectra. In fact, there is no known way. Instead, we will present an alternative category of spectra, which has a well behaved smash product:

Symmetric spectra

Let Σ_n be the symmetric group on n letters. If we define $S^m = S^1 \wedge ... \wedge S^1$, then we have a Σ_m-action by permuting the S^1's. Write $\Sigma_m \times \Sigma_n \subset \Sigma_{m+n}$ for the subgroup that permutes the first m letters and the last n letters, separately.

Definition 4.39. A *symmetric spectrum* X is a sequence $\{X_n\}$ of based Σ_n-spaces, with structure maps

$$\sigma_{m,n} : S^m \wedge X_n \to X_{m+n}$$

which are $\Sigma_m \times \Sigma_n$-equivariant and make the square

$$\begin{tikzcd} S^k \wedge S^m \wedge X_n \arrow[r, "\Sigma^k \sigma_{m,n}"] \arrow[d, "\mathsf{id}"'] & S^k \wedge X_{m+n} \arrow[d, "\sigma_{k,m+n}"] \\ S^{k+m} \wedge X_n \arrow[r, "\sigma_{k+m,n}"] & X_{k+m+n} \end{tikzcd}$$

commute.

Symmetric spectra form a monoidal category. We define the smash product as follows. First define a 'pre-smash product' $X \wedge_0 Y$ by

$$(X \wedge_0 Y)_n = \bigvee_{i+j=n} (\Sigma_n)_+ \wedge_{\Sigma_i \times \Sigma_j} (X_i \wedge Y_j).$$

Here Σ_n is considered as a discrete space. By $\wedge_{\Sigma_i \times \Sigma_j}$ we mean take the smash product and then quotient out by $\Sigma_i \times \Sigma_j$.

As a space, $(\Sigma_n)_+ \wedge_{\Sigma_i \times \Sigma_j} (X_i \wedge Y_j)$ is $\bigvee_{\binom{n}{i}} X_i \wedge Y_j$. So $\wedge_0$ is associative up to natural isomorphism.

The structure maps of X give us a map $\alpha_X : S \wedge_0 X \to X$ by

$$\bigvee_{i+j=n} (\Sigma_n)_+ \wedge_{\Sigma_i \times \Sigma_j} (S^i \wedge X_j) \xrightarrow{\vee \sigma_{i,j}} X_n.$$

Define the smash product $X \wedge Y$ of symmetric spectra X and Y as the coequaliser of

$$X \wedge_0 S \wedge_0 Y \underset{\mathsf{id}_X \wedge_0 \alpha_Y}{\overset{\alpha_X \wedge_0 \mathsf{id}_Y}{\rightrightarrows}} X \wedge_0 Y.$$

4.3 Some Stable Homotopy Theory

In this section, we will quickly introduce some relevant stable theory. A general idea in mathematics is to call a phenomenon *stable* if it occurs in the same way for all sufficiently large dimensions. Otherwise, it is *unstable*.

It turns out that many problems that were originally thought to be unstable can in fact be reduced to stable problems, which are often easier to solve.

Definition 4.40. The homotopy class of (based) maps between (spaces) X and Y is stable if it is in bijection with $[\Sigma^n X, \Sigma^n Y]$ for arbitrary $n \geq 0$.

4.3.1 The Freudenthal Suspension Theorem

The field of stable homotopy theory arose from the Freudenthal suspension theorem. It shows that homotopy groups of spheres are stable.

Theorem 4.41 (the Freudenthal suspension theorem)**.** *The suspension map*

$$\pi_n(S^m) \to \pi_{n+1}(S^{m+1})$$

is an isomorphism for $n < 2m - 1$ and a surjection for $n = 2m - 1$. More generally, this holds for the suspension $\pi_n(X) \to \pi_{n+1}(\Sigma X)$ whenever X is an $(m-1)$-connected CW complex.

A proof can be found in [9, corollary 4.24].

In particular, if X is a CW complex with bottom cell in dimension m, then the n-th homotopy group of X is stable, provided $n < 2m - 1$. In the next section we will prove the dual statement: if X is a finite CW complex with top cells in dimension n, then the m-th cohomotopy group† of X is stable, provided $n < 2m - 1$.

†While homotopy groups consist of maps out of spheres, cohomotopy groups are the dual: maps into

4.3.2 A Stability Theorem for Cohomotopy Groups

We want to prove a stability result for cohomotopy groups of finite CW complexes. To do this, we will start with a map $f : X \to Y$, add a cell D^k to X, and look at the obstructions to extensions $X \cup_\phi D^k \to Y$ of f:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook] & Y \\ X \cup_\phi D^m \arrow[ur, dashed] & \end{tikzcd}$$

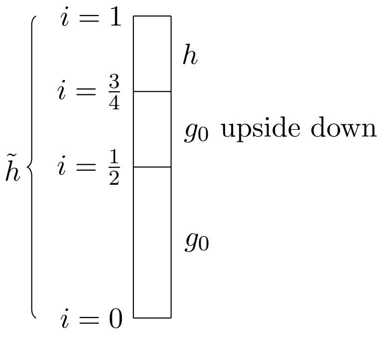
An extension $X \cup_\phi D^m \to Y$ exists if and only if $S^{m-1} \xrightarrow{\phi} X \xrightarrow{f} Y$ is null-homotopic. Why? We need to define a map $D^m \to Y$ that agrees with f on the boundary. But this is equivalent to defining a homotopy from a map $S^{m-1} \to Y$ to a constant map.

Fix one extension $g_0 : D^m \to Y$. Build a map $h : S^m \to Y$ out of another extension $g : D^m \to Y$ as follows: Define h on the northern hemisphere D^m by g and on the southern hemisphere by g_0. We know h is well defined, since g and g_0 agree on the equator S^{m-1}. It is clear that this construction is invariant under homotopy.

This construction gives a bijection Φ between the homotopy classes of extensions and elements $[h] \in \pi_m(Y)$. It is obviously injective. To see it is surjective, consider $h : S^m \to Y$ as a map $S^{m-1} \times I \to Y$ which is constant on $\{0\} \times S^{m-1}$ and on $\{1\} \times S^{m-1}$. We can also consider an extension $g : D^m \to Y$ as a map $S^{m-1} \times I \to Y$ which is constant on $\{0\} \times S^{m-1}$ and equal to f on $\{1\} \times S^{m-1}$.

Define $\tilde{h} : S^{m-1} \times I \to Y$ by

$$\tilde{h}(s,i) = \begin{cases} g_0(s,2i) & \text{if } i \in [0,0.5], \\ g_0\big(s, 1-4(i-0.5)\big) & \text{if } i \in [0.5,0.75], \\ h\big(s,4(i-0.75)\big) & \text{if } i \in [0.75,1]. \end{cases}$$



We know that $\tilde{h}$ is homotopic to h. But the northern hemisphere is an extension $D^m \to Y$ of f and the southern hemisphere is the extension g_0. So $[h]$ is in the image of Φ. Therefore, we have the following lemma:

spheres. We define the m-th cohomotopy set of X to be the set of homotopy classes $\pi^m(X) = [X, S^m]$. It turns out that we can give $\pi^m(X)$ a group structure, provided that X is a CW complex of dimension at most $2m-2$.

Lemma 4.42. *Given $f : X \to Y$, if f extends to a map $X \cup D^k \to Y$, then the set of homotopy classes of extensions is in bijection with $\pi_k(Y)$.*

Theorem 4.43. *Let X be an n-dimensional CW complex. If $n < 2m-1$, then $[X, S^m]$ is stable.*

Proof. We will induct on the cells of X. Let $\tilde{X}$ be an arbitrary subcomplex of X.

Base case: If $\tilde{X}$ is a single cell, then it has dimension $\tilde{n} \leq n$. By the Freudenthal suspension theorem, $[\tilde{X}, S^m]$ is stable.

Step case: Assume that both $[\tilde{X}, S^m]$ and the obstruction to extending f are stable. Add a cell D^k to $\tilde{X}$ with attaching map ϕ. We want to show that $[\tilde{X} \cup_\phi D^k, S^m]$ is stable. Each class $[f'] \in [\tilde{X} \cup_\phi D^k, S^m]$ is an extension of some class $[f] \in [\tilde{X}, S^m]$. Moreover, the number of extensions of $[f]$ is in bijection with $\pi_k(S^m)$.

Now, each class $[g'] \in [\Sigma^n \tilde{X} \cup_{\Sigma\phi} D^{k+n}, S^{m+n}]$ is an extension of some class $[g] \in [\Sigma^n \tilde{X}, S^{m+n}]$. The number of extensions of $[g]$ is in bijection with $\pi_{k+n}(S^{m+n})$. But we know that $\pi_k(S^m) \cong \pi_{k+n}(S^{m+n})$ by the Freudenthal suspension theorem and that $[\tilde{X}, S^m]$ is in bijection with $[\Sigma^n \tilde{X}, S^{m+n}]$. Thus, $[\tilde{X} \cup_\phi D^k, S^m]$ is in bijection with $[\Sigma^n \tilde{X} \cup_{\Sigma\phi} D^{k+n}, S^{m+n}]$. □

Note that we only used the Freudenthal suspension theorem to conclude that $\pi_k(S^m)$ was stable for $k \leq n < 2m-1$. Thus, the theorem also holds if we replace S^m with any $(m-1)$-connected CW complex.

4.3.3 Maps of Suspension Spectra in the Stable Range

Recall Proposition 4.20: we have a natural 1-to-1 correspondence

$$[\Sigma^\infty X, E]_r \leftrightarrow \operatorname*{colim}_{n\to\infty} [\Sigma^{n+r} X, E_n],$$

where X is a finite CW complex and E a spectrum.

Now, if it turns out that the homotopy classes $[\Sigma^{n+r} X, E_n]$ stabilise, that is,

$$\operatorname*{colim}_{n\to\infty} [\Sigma^{n+r} X, E_n] = [\Sigma^{N+r} X, E_N]$$

for some large N, then maps of spectra $[\Sigma^\infty X, E]_r$ correspond to maps of spaces.

Suppose that E was also a suspension spectrum $\Sigma^\infty Y$. By theorem 4.43, the sequence $[\Sigma^{n+r} X, \Sigma^n Y]$ is stable if $\Sigma^r X$ is an n-dimensional CW complex; Y is $(m-1)$-connected; and $n < 2m-1$. Re-arranging this gives us the following corollary.

Corollary 4.44. *Let X, Y be CW complexes. Suppose X is finite with top cell in dimension n and Y is $(m-1)$-connected. If $n + r < 2m - 1$ then maps between spectra $[\Sigma^\infty X, \Sigma^\infty Y]_r$ correspond to maps between spaces $[\Sigma^r X, Y]$.*

When we prove the vector fields problem, we will move from spaces into the stable homotopy category via the functor Σ^∞. This will allow us to exploit some stable properties. But to conclude the proof, we will need to move back into the category of spaces. The above proposition will provide this link back. In particular, we now know that if we are in the stable range, a space is (co)reducible if and only if its suspension spectrum is (co)reducible. (We have not yet defined (co)reducibility of spectrum. We will do so in the next chapter, but intuitively it means that the top or bottom stable cell splits.)

Chapter 5

Duality

In this chapter, we assume that there exists a commutative and associative tensor product, which we call the smash product $\wedge$, in the stable homotopy category. Moreover, this smash product is compatible with the previously defined smash product between a space and a spectra. So given a spectrum E and a CW complex X,

$$E \wedge X \cong E \wedge \Sigma^\infty X$$

where $X \wedge E$ is the spectrum defined by $(E \wedge X)_n = E_n \wedge X$ with structure maps $S^1 \wedge E_n \wedge X \xrightarrow{\sigma_n \wedge \mathsf{id}_X} E_{n+1} \wedge X$.

Thus, the smash product $\wedge$ makes the stable homotopy category into a symmetric monoidal category. See [2, Part III, §4] for the construction of $\wedge$ and proof of these facts.

Further, we assume that the smash product $\wedge$ preserves cofibre sequences and the sphere spectrum S^0 is the unit object of $\wedge$. We assume that the function spectrum $F(X, Y)$, defined below, is the internal hom object in the stable homotopy category. So there is an adjunction

$$[W \wedge X, Y] \cong [W, F(X, Y)] \tag{5.1}$$

5.1 Spanier-Whitehead Duality

5.1.1 Function Spectra

Definition 5.1. Let X and Y be spectra and W a CW complex. The functor sending a CW complex W to the morphisms from $X \wedge W$ to Y in the stable homotopy category

$$W \mapsto [X \wedge W, Y]$$

is a generalised reduced cohomology theory. Brown's representability theorem then implies that it is represented by an object in the stable homotopy category. We define this object to be the *function spectrum* $F(X, Y)$.

5.1.2 Dual Objects in the Stable Homotopy Category

Definition 5.2. The *Spanier-Whitehad dual* of a spectrum E is the dual object[†] of E in the stable homotopy category, if it exists. We denote the dual by DE.

Proposition 5.3. *If the dual of X exists, it is $F(X, S^0)$.*

This is a result in category theory, which holds for any closed symmetric monoidal category.

Proposition 5.4. *D is a contravariant functor on the subcategory of dualisable objects.*

Proof. We need to show that $[DX_2, DX_1] \cong [X_1, X_2]$ for dualisable objects X_1, X_2. We use the adjoint property to get

$$[DX_2, DX_1] \cong [S^0, X_2 \wedge DX_1] \cong [S^0 \wedge X_1, X_2] \cong [X_1, X_2]. \qquad \square$$

Lemma 5.5. *If $A \to X \to Y = X \cup CA$ is a cofibre sequence such that A and X are dualisable, then Y is dualisable and the sequence $DY \to DX \to DA$ is a fibre (or equivalently cofibre[†]) sequence.*

We omit the proof. See [25, §8].

Proposition 5.6. $\pi_r F(Y, Z) = [Y, Z]_r$

Proof. By Proposition 4.20, $\pi_r F(Y, Z) \cong [S^0, F(Y, Z)]_r$. But $[S^0, F(Y, Z)]_r \cong [S^0 \wedge Y, Z]_r$ by the adjunction 5.1 and $[S^0 \wedge Y, Z]_r \cong [Y, Z]_r$ since S^0 is the unit of $\wedge$. $\square$

Proposition 5.7. $DY \wedge Z \cong F(Y, Z)$

Proof. We need to show that $[X, DY \wedge Z]$ and $[X, F(Y, Z)] \cong [X \wedge Y, Z]$ are isomorphic, for all spectra X. This is a standard result in category theory, see Theorem A.8 in the appendix. $\square$

5.1.3 Cell Dimensions are Inverted when Taking Duals

If $Y = S^0$ and $X = S^n$, then $F(X, Y)$ represents the cohomology

$$E^k(W) = \lim_i [\Sigma^i (W \wedge S^n), S^{k+i}] \cong \mathbb{S}^{k-n}(W),$$

where $\mathbb{S}^*$ is the cohomology theory represented by the sphere spectrum. This implies that DS^n and S^{-n} are isomorphic in the stable homotopy category, since they represent the same cohomology. So the dual of S^n is S^{-n}.

So now we know that the dimension of the spheres are inverted when we take duals. Since CW-spectra are made up of spheres, we should be able to show the cells of any CW spectrum are also inverted when taking duals.

[†]See the appendix for the definition of a dual object, in category theory.

[†]In the stable homotopy category, every cofibre sequence is a fibre sequence, and visa versa. See [25, §7].

Suppose that X is an n-dimensional CW spectra. Then X^n/X^{n-1} dualises to the $(-n)$-skeleton of DX. By induction, X^n/X^{k-1} dualises to the $(-k)$-skeleton of DX. Moreover, we have a cofibre sequence

$$X^n/X^{k-1} \to X^n/X^k \to X^k/X^{k-1} = \bigvee S^k$$

Dualising, we get

$$D(X^n/X^{k-1}) \leftarrow D(X^n/X^k) \leftarrow \bigvee S^{-k},$$

by Lemma 5.5. It turns out that this is the attaching map for the $(-k)$-cells. Why? It is a cofibre sequence where $D(X^n/X^k)$ only has cells from dimension $-n$ to dimension $-k-1$ and $D(X^n/X^{k-1})$ has cells from dimension $-n$ to dimension $-k$.

We have proved the following theorem:

Theorem 5.8. *Let X be a finite CW spectrum. Then the dual of X exists.*

5.1.4 S-reducible and S-coreducible

Definition 5.9. Let X be a $(n-1)$-dimensional CW complex and suppose Y constructed from X by attaching an n-cell via $f : S^{n-1} \to X$. Let $q : \Sigma^\infty Y \to \Sigma^\infty S^n$ be the map collapsing $\Sigma^\infty X$ to a point. Then Y is *S-reducible* if there exists a map $f : \Sigma^\infty S^n \to \Sigma^\infty Y$ such that post-composition with q

$$\Sigma^\infty S^n \xrightarrow{f} \Sigma^\infty Y \xrightarrow{q} \Sigma^\infty S^n$$

has degree 1. That is, the map $\mathbb{Z} = \pi_n(\Sigma^\infty S^n) \xrightarrow{q_* f_*} \pi_n(\Sigma^\infty S^n)$ is multiplication by 1.

We can think of Y being S-reducible if the attaching map of the stable n-cell of $\Sigma^\infty Y$ is trivial. Equivalently, the r-fold suspension of the attaching map of the n-cell of Y is trivial, for sufficiently large values of r.

Definition 5.10. Let Y be a CW complex such that there is a single cell in dimension n, and all cells (except perhaps the base point 0-cell) have dimension greater than n. Define $i : \Sigma^\infty S^n \hookrightarrow \Sigma^\infty Y$ to be the inclusion map of the corresponding stable cell in $\Sigma^\infty Y$. Then Y is *S-coreducible* if there exists a map $f : \Sigma^\infty Y \to \Sigma^\infty S^n$ such that pre-composition with i

$$\Sigma^\infty S^n \xrightarrow{i} \Sigma^\infty Y \xrightarrow{f} \Sigma^\infty S^n$$

has degree 1.

S-reducibility and S-coreducibility are weaker conditions than reducibility and coreducibility. However, if we are in the stable range, then they are equivalent properties. Since the Spanier-Whitehead dual inverts the dimensions of the stable cells and is a contravariant functor, we have that:

Proposition 5.11. *The dual DX is S-coreducible if and only if X is S-reducible, and visa versa.*

5.2 Alexander Duality

Alexander duality is a standard result relating the homology of X with the cohomology of $S^n - X$:

Theorem 5.12. *Let X be a finite CW complex embedded into S^n, such that $S^n - X$ is homotopy equivalent to a finite CW complex. Then there is an isomorphism*

$$\widetilde{H}_r(X;\mathbb{Z}) \xrightarrow{\cong} \widetilde{H}^{n-1-r}(S^n - X;\mathbb{Z}).$$

This is a specialisation of corollary 3.45 of [9]. We omit the proof. A corollary is the Alexander duality for spectra:

Theorem 5.13. *Let X be as in Theorem 5.12. Then there is an isomorphism*

$$DX \cong \Sigma^{-(n-1)}(S^n - X)$$

in the stable homotopy category.

We delay the proof until the end of this section. It is not too difficult to prove standard Alexander duality from Alexander duality for spectra: We have an isomorphism $D(S^n - X) \to \Sigma^{-(n-1)}X$, since we can interchange X and $S^n - X$. Then by smashing with the Eilenberg-MacLane spectrum $H\mathbb{Z}$ and taking the induced map on the $(r+1-n)$-th homotopy group, we get

$$\pi_{r+1-n}\left(D(S^n - X) \wedge H\mathbb{Z}\right) \xrightarrow{\cong} \pi_{r+1-n}\left(\Sigma^{-(n-1)}X \wedge H\mathbb{Z}\right).$$

But we have on the left hand side

$$\pi_{r+1-n}\left(D(S^n - X) \wedge H\mathbb{Z}\right) \cong \pi_{r+1-n}F(S^n - X, H\mathbb{Z}) \cong [S^n - X, H\mathbb{Z}]_{r+1-n} \cong \widetilde{H}^{n-1-r}(S^n - X;\mathbb{Z}),$$

where the first isomorphism follows from Proposition 5.7 and the second from Proposition 5.6. On the right we have

$$\pi_{r+1-n}\left(\Sigma^{-(n-1)}X \wedge H\mathbb{Z}\right) \cong \pi_r\left(X \wedge H\mathbb{Z}\right) \cong \widetilde{H}_r(X;\mathbb{Z}),$$

where the final isomorphism is Proposition 4.23.

So Theorems 5.12 and 5.13 are equivalent.

Lemma 5.14. *Suppose Y is a finite CW spectrum with $[Y, H\mathbb{Z}]_r = 0$ for all $r \in \mathbb{Z}$. Then $Y \cong *$ in the stable homotopy category.*

Proof. This follows from the Hurewicz theorem, which holds for spectra[†]. □

[†]Alternatively, here is a proof that doesn't rely of the Hurewicz theorem:We need only show that the homotopy groups of Y are trivial. Choose N such that the N-th component Y_N contains representatives for every stable cell of Y. We can do this since Y has a finite number of stable cells. Let K be a finite subcomplex of Y_N containing all the representatives. By passing to $\Sigma K \subset Y_{N+1}$ if necessary, we may assume that K is simply connected.

Proof of Theorem 5.13. Let A be a finite CW complex such that A is homotopy equivalent to $S^n - X$.

First we define a natural map

$$\phi : \Sigma^{-(n-1)} A \to DX$$

then we will show it is an isomorphism in the stable homotopy category.

Suppose, for the moment, that, when A and X are embedded in S^n, there is no point in A which is antipodal to a point in X. Then, for any tuple $(a, x) \in A \times X$, there is a unique great circle of S^n containing both a and x. Let $\gamma_{a,x} : [0, 1] \to S^n$ be the arc of length one from a to x along this great circle. We have a map

$$\begin{aligned} A * X &\to S^n, \\ (a, x, t) &\mapsto \gamma_{a,x}(t). \end{aligned}$$

Note that $A * X$ is homotopy equivalent to $\Sigma(A \wedge X)$. Thus, we get a map $\Sigma(A \wedge X) \to S^n$. This gives a map $\Sigma^{-(n-1)} A \wedge X \to S^0$ in the stable homotopy category, and consequently

$$\phi : \Sigma^{-(n-1)} A \to F(X, S^0) \cong DX,$$

as required.

Now, if there is a pair of antipodal points, one in A and one in X, the construction of ϕ is more difficult. We need to use the Hopf construction, which takes a map $W \times Y \to Z$ and builds a map $W * Y \to \Sigma Z$. To define the Hopf construction, note that the join is the pushout

$$\begin{tikzcd} W * Y & C'W \times Y \arrow[l, dashed] \\ W \times C'Y \arrow[u, dashed] & W \times Y \arrow[l, hook', "i_Y"] \arrow[u, hook, "i_W"'] \end{tikzcd}$$

The suspension spectrum of K has n-th component $\Sigma^n K$. The inverse suspension shifts indices down, so $\Sigma^{-n} K$ has n-th component K. Thus, $\Sigma^{-n} K$ is a cofinal subspectrum of Y and so $\Sigma^{-n} K$ and Y are isomorphic in the stable homotopy category. But we know

$$0 = [Y, H\mathbb{Z}]_r \cong [\Sigma^{-n} K, H\mathbb{Z}]_r \cong [K, H\mathbb{Z}]_{r-n} \cong \widetilde{H}^{n-r}(K; \mathbb{Z}).$$

So all the cohomology groups of K are zero.

We can use the universal coefficient theorem to conclude the homology groups of K must also be zero: Specifically, we know that $\mathrm{Hom}(H_n(X), \mathbb{Z}) \cong 0$ and $\mathrm{Ext}(H_n(X), \mathbb{Z}) \cong 0$ for all $n > 0$. Also, $H_n(X)$ is a finitely generated abelian group by [9, Proposition 3F.12], so $H_n(X) \cong \mathbb{Z}^r \oplus \mathbb{Z}_{p_1}^{m_1} \oplus ... \oplus \mathbb{Z}_{p_s}^{m_s}$. Then

$$\mathbb{Z}^r \cong \mathrm{Hom}(H_n(X), \mathbb{Z})$$

implies $r = 0$ and

$$\mathbb{Z}_{p_1}^{m_1} \oplus ... \oplus \mathbb{Z}_{p_s}^{m_s} \cong \mathrm{Ext}(H_n(X), \mathbb{Z})$$

implies $s = 0$. So $H_n(X) = 0$ for $n > 0$. Finally, the Hurewicz theorem (for spaces) then implies that all the homotopy groups of K are zero: We have assume K is simply connected. So we can apply the Hurewicz theorem to find that $\pi_2(K) \to H_2(K) \cong 0$ is an isomorphism. By induction, $\pi_n(K) \to H_n(K) \cong 0$ is an isomorphism.

where C' is the unreduced cone. There are obvious maps

$$W \times C'Y \to C'Z \to \Sigma Z \text{ and } C'W \times Y \to C'Z \to \Sigma Z,$$

which commute with the inclusions i_Y, i_W. Therefore, by the universal property of pushouts, we have a map $W * Y \to \Sigma Z$.

Returning to the problem at hand, suppose there is a pair of antipodal points. By changing A if necessary, we can assume that there is a point p in S^n not in $X \cup A$. Defining $\mathrm{St} : S^n - \{p\} \to \mathbb{R}^n$ to be the stereographic projection at point p we get a map

$$\begin{aligned} X \times A &\to S^{n-1}, \\ (x, a) &\mapsto \frac{\mathrm{St}(x) - \mathrm{St}(a)}{|\mathrm{St}(x) - \mathrm{St}(a)|}, \end{aligned}$$

as required. The Hopf construction on this map gives us ϕ.

Consider the cofibre sequence of ϕ

$$\Sigma^{-(n-1)} A \xrightarrow{\phi} DX \hookrightarrow DX \cup C(\Sigma^{-(n-1)} A) = Y.$$

Since cofibre sequences are fibre sequences, they induce long exact sequences of homotopy groups. If $\pi_r(Y) \cong 0$ then ϕ induces an isomorphism on homotopy groups for all r. Consequently, ϕ is an isomorphism in the stable homotopy category.

We know that taking Spanier-Whitehead duals and the smash product preserve cofibre sequences, so both

$$DY \to X \to D\Sigma^{-(n-1)} A \text{ and } DY \wedge H\mathbb{Z} \to X \wedge H\mathbb{Z} \to D\left(\Sigma^{-(n-1)} A\right) \wedge H\mathbb{Z}$$

are cofibre sequences. Consider the exact sequence of homotopy groups induced by the last cofibre sequence. We have that $\pi_r(X \wedge H\mathbb{Z}) \cong \widetilde{H}_r(X, \mathbb{Z})$ by Proposition 4.23. Additionally,

$$\begin{aligned} \pi_r \left(D\left(\Sigma^{-(n-1)} A\right) \wedge H\mathbb{Z} \right) &\cong \pi_r F(\Sigma^{-(n-1)} A, H\mathbb{Z}) \\ &\cong [\Sigma^{-(n-1)} A, H\mathbb{Z}]_r = [A, H\mathbb{Z}]_{r-(n-1)} \\ &\cong \widetilde{H}^{n-1-r}(A, \mathbb{Z}), \end{aligned}$$

using Propositions 5.7 and 5.6. Now, it turns out that the isomorphism in Theorem 5.12 is the map $\widetilde{H}_r(X, \mathbb{Z}) \to \widetilde{H}^{n-1-r}(A, \mathbb{Z})$ in this long exact sequence. Therefore,

$$0 \cong \pi_r \left(DY \wedge H\mathbb{Z} \right) \cong \pi_r F(Y, H\mathbb{Z}) = [Y, H\mathbb{Z}]_r,$$

for all r. By Lemma 5.14, $Y \cong *$ and consequently $\pi_r(Y) \cong 0$ as required. □

5.3 Atiyah Duality

We now examine the Spanier-Whitehead duals of manifolds and the duals of Thom spaces of vector bundles over manifolds. This is due to Atiyah [3].

We know that $\mathrm{Th}(V \oplus \epsilon^n) = \Sigma^n \mathrm{Th}(V)$, where V is a vector bundle over X. Now that we have an inverse suspension, we should be able to extend this, at least in the stable homotopy category. Define the Thom spectrum of any element $V - \epsilon^n$ of $K_{\mathbb{F}}(X)$ by

$$\mathrm{Th}(V - \epsilon^n) = \Sigma^{-n}\Sigma^\infty \mathrm{Th}(V).$$

in the stable homotopy category.

Let M be a n-dimensional differentiable manifold. Suppose we have an embedding $M \to \mathbb{R}^m$. Then we can define the normal bundle $N_{M,\mathbb{R}^m}$ whose fibre at p is the $(m-n)$-dimensional space of vectors in $\mathbb{R}^m$ normal to p. The sum of the normal and tangent bundles is trivial:

$$T_M \oplus N_{M,\mathbb{R}^m} \cong \epsilon^m.$$

By rearranging this equation, we have $Th(-T_M) \cong \Sigma^{-m}\mathrm{Th}(N_{M,\mathbb{R}^m})$. This doesn't depend on the choice of embedding since any two embeddings are stably equivalent.

Definition 5.15. A vector bundle $p : E \to B$ is *differentiable* if E and B are differentiable manifolds, p is a differentiable map and the local trivialisations of p are diffeomorphisms.

Atiyah duality is a powerful result. Assuming the Thom isomorphism theorem, Poincaré duality (for generalised cohomology) is an easy corollary. (See [25, §10] for the proof.)

Theorem 5.16. *Suppose M is a compact, differentiable manifold. If M has a boundary ∂M, then*

$$D(M/\partial M) \cong \mathrm{Th}(-T_M)$$

Otherwise, if M doesn't have a boundary and $E \to M$ is a differentiable vector bundle, then

$$D(\mathrm{Th}E) \cong \mathrm{Th}(-T_M - E).$$

An easy corollary is that $D(M_+) \cong \mathrm{Th}(-T_M)$ for M a compact, smooth manifold without a boundary, since $\mathrm{Th}(B, \epsilon^0) = B_+$ for any space B.

The second statement also holds for a real (not necessarily differentiable) vector bundle E. To see this, we need to know about the universal bundle $E_n(\mathbb{R}^\infty)$ over the Grassmannian $G_n(\mathbb{R}^\infty)$, the space of all n-dimensional linear subspaces of $\mathbb{R}^\infty$. See [10, §1.2]. Let $f : M \to G_n$ be the map inducing E–that is, $f^*(E_n) = E$. We may approximate f by a differentiable map, which will induce a differentiable bundle isomorphic to E. Then, the corresponding Thom spectra will also be isomorphic.

Proof. Suppose M has a boundary ∂M. We can embed M into D^m, for m sufficiently large, so that ∂M is in the boundary S^{m-1}, with M transverse to S^{m-1}. (See Lemma 3.1 of [3] for details.) Additionally, we may assume that M is a finite CW complex (see [15]) and this embedding is cellular.

By considering the relevant cofibre sequence, $M/\partial M$ is homotopy equivalent to

$$Y = M \cup C'(\partial X) \subset D^m \cup C'S^{m-1} \cong S^m$$

where C' is the unreduced cone. We can apply Alexander duality to Y to get

$$D(M/\partial M) \cong \Sigma^{-(m-1)}(S^m - Y).$$

Let N be a tubular neighbourhood (c.f. chapter 3 of [16]) of M. Then $N' = N \cup C'(N \cap S^{m-1})$ deformation retracts onto Y. So $S^m - Y \simeq S^m - N'$. Since the tip of the cone $C'S^{m-1}$ is in $C'(N \cap S^{m-1})$, we have that $C'S^{m-1} - C'(N \cap S^{m-1})$ deformation retracts to $S^{m-1} - N \cap S^{m-1}$. Therefore, $S^m - N'$ deformation retracts to $D^m - N$ and consequently

$$D(M/\partial M) \cong \Sigma^{-(m-1)}(D^m - N).$$

As D^m is contractible, $D^m - N \to D^m \to \Sigma(D^m - N)$ is a cofibre sequence. So,

$$\Sigma(D^m - N) \simeq D^m/(D^m - N).$$

Since N is open, $D^m/(D^m - N) = \overline{N}/\partial N$, where $\overline{N}$ is the closure of N.

We may choose N such that $\overline{N}$ and ∂N are identified with the disk bundle and the sphere bundle of N_{M,D^m} respectively. Thus, $\overline{N}/\partial N \cong \mathrm{Th}(N_{M,D^m})$ and

$$D(M/\partial M) \cong \Sigma^{-m}\Sigma(D^m - N) \cong \Sigma^{-m}\mathrm{Th}(N_{M,D^m}) \cong \mathrm{Th}(-T_M).$$

This proves the first statement. The second statement is a corollary: The disk bundle $\mathcal{D}$ of E is a compact differentiable manifold with boundary the sphere bundle of E. So

$$D(\mathrm{Th}E) = D(\mathcal{D}/\partial\mathcal{D}) \cong \mathrm{Th}(-T_{\mathcal{D}}).$$

Let $p : \mathcal{D} \to M$ be the bundle map of $\mathcal{D}$. With some work, we find $T_{\mathcal{D}}$ and $p^*(T_M \oplus E)$ represent the same class in $K(\mathcal{D})$. So their Thom spectra are isomorphic in the stable homotopy category $\mathrm{Th}(-T_{\mathcal{D}}) \cong \mathrm{Th}\left(-p^*(T_M \oplus E)\right)$ by Proposition 4.34. Since p is a homotopy equivalence, we conclude $\mathrm{Th}\left(-p^*(T_M \oplus E)\right) \cong \mathrm{Th}(-T_M - E)$. $\square$

5.3.1 Dual of Stunted Projective Spaces

Recall that $\mathbb{R}\mathrm{P}_k^{n+k} = \mathrm{Th}(\mathbb{R}\mathrm{P}^n, k\xi)$ where ξ is the canonical line bundle on $\mathbb{R}\mathrm{P}^n$.

Theorem 5.17. *The Spanier-Whitehead dual of* $\mathbb{R}\mathrm{P}_k^{n+k}$ *is* $\Sigma\mathbb{R}\mathrm{P}_{-(n+k+1)}^{-(k+1)}$.

Proof. We will show in chapter 7 that $K_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n)$ is generated by ξ. So there is only one class in $K_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n)$ for each (virtual) dimension. Since the tangent space $T_{\mathbb{R}\mathrm{P}^n}$ is n dimensional, $(n+1)\xi - \epsilon^1$ represents the same class as $T_{\mathbb{R}\mathrm{P}^n}$. Therefore, by Atiyah duality,

$$\begin{aligned} D\mathbb{R}\mathrm{P}_k^{n+k} &\cong \mathrm{Th}(\mathbb{R}\mathrm{P}^n, -T_{\mathbb{R}\mathrm{P}^n} - k\xi) \\ &\cong \mathrm{Th}(\mathbb{R}\mathrm{P}^n, -(n+1)\xi + \epsilon^1 - k\xi) \\ &\cong \Sigma\mathrm{Th}(\mathbb{R}\mathrm{P}^n, -(n+k+1) \\ &\cong \Sigma\mathbb{R}\mathrm{P}_{-(n+k+1)}^{-(k+1)}. \end{aligned}$$

$\square$

Chapter 6

Spectral Sequences

Spectral sequences are a powerful computational tool in algebraic topology. In chapter 7, the Atiyah Hirzebruch spectral sequence will be essential in our computations of the K theory of stunted projective spaces.

6.1 A General Formulation

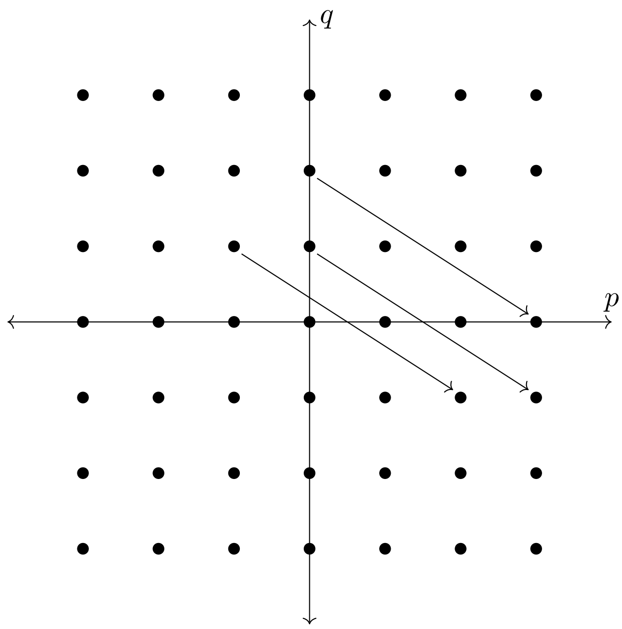
We take the approach given in [18, chapter 2] and [11], although spectral sequences exist in more general environments (for example, abelian categories). We also restrict our attention to spectral sequences of cohomological type.

Definition 6.1. Let R be a ring. A *spectral sequence* is a collection of R-modules $\{E_r^{p,q}\}_{r\in\mathbb{N}_{>0}}^{p,q\in\mathbb{Z}}$ and differentials $d_r : E_r^{*,*} \to E_r^{*+r,*-(r-1)}$ such that

1. $d_r \circ d_r = 0$,
2. $E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*}, d_r) := \text{Ker}\, d_r : E_r^{p,q} \to E_r^{p+r,q-(r-1)} \Big/ \text{Im}\, d_r : E_r^{p-r,q+(r-1)} \to E_r^{p,q}$

for all r, p, q.

A spectral sequence can be thought of as a book. For fixed r, the modules $E_r^{p,q}$ are called the r-th page. (Often the r-th page is denoted simply by E_r.) On each page, the modules $E_r^{*,*}$ form an integral lattice in the Cartesian plane. For example, the 3rd page of a spectral sequence is below.



To avoid cluttering the lattice, we have omitted many of the differentials.

It does not really matter which page the spectral sequence starts on. In practise, spectral sequences are often defined for $r \geq 0$, $r \geq 1$ or $r \geq 2$.

6.1.1 Exact and Derived Couples

Often, spectral sequences are built from algebraic objects called exact couples. This is the case with the Atiyah Hirzebruch spectral sequence.

Definition 6.2. An *exact couple* is a pair F, E of abelian groups and an exact diagram

$$\begin{tikzcd} F \arrow[dr, "j"'] & & F \arrow[ll, "i"'] \\ & E. \arrow[ur, "k"'] & \end{tikzcd}$$

Note that $d = jk$ is a differential, since $d^2 = jkjk = j0k = 0$. So we can consider the homology $\operatorname{Ker} jk / \operatorname{Im} jk$.

Definition 6.3. Given an exact couple (F, E, i, j, k), the associated *derived couple* is the pair $F' = i(F) \subset F$ and $E' = \operatorname{Ker} jk / \operatorname{Im} jk$ of abelian groups and morphisms

$$\begin{tikzcd} F' \arrow[dr, "j'"'] & & F' \arrow[ll, "i'"'] \\ & E. \arrow[ur, "k'"'] & \end{tikzcd}$$

defined below.

1. The morphism $i' = i|_{F'}$ is the restriction of i to its image.

2. We can write an arbitrary element of F' as $i(f)$ for $f \in F$. So define j' by $j'\big(i(f)\big) = [j(f)] \in E'$. This is well defined: firstly $j(f) \in \operatorname{Ker} jk$ since $jkj(f) = 0$; secondly, if $i(f_1) = i(f_2)$ then $f_1 - f_2 \in \operatorname{Ker} i = \operatorname{Im} k$, do $j(f_1) - j(f_2) \in \operatorname{Im} jk$.

3. Define $k'[e] = ke$. Since $e \in \operatorname{Ker} jk$, we have $ke \in A' = \operatorname{Im} i = \operatorname{Ker} j$, so this definition makes sense. It is well defined since $[e_1] - [e_2] = 0 \in E'$ implies $e_1 - e_2 \in \operatorname{Im} jk \subset \operatorname{Im} j = \operatorname{Ker} k$.

Lemma 6.4. *A derived couple is exact.*

This is an exercise in diagram chasing; see [11, lemma 1.1] for a proof.

By iterating the process of forming derived couple, we get a sequence $E, E', E'', ...$ with differentials $d, d', ...$ We call E the zeroth derived couple of E; E' the first derived couple of E; E'' the second, and so on. These can often be combined to form a spectral sequence. We will illustrate this with an example.

Let $\widetilde{E}$ be a reduced cohomology theory. Suppose X is a CW complex and we filter X by an increasing chain of subcomplexes

$$\emptyset \subset X^0 \subset ... \subset X^n \subset ... \subset X.$$

Note that this filtration does not necessarily need to be the skeletal filtration, but (X^k, X^{k-1}) must be a CW pair, so that we can form long exact sequences of cohomology groups (c.f. Definition 1.28). These long exact sequences fit into a diagram

$$\begin{tikzcd}
\widetilde{E}^*(X^0) \arrow[dr, dashed, "j"'] & & \widetilde{E}^*(X^1) \arrow[ll, "i"'] \arrow[dr, dashed, "j"'] & & \widetilde{E}^*(X^2) \arrow[ll, "i"'] \arrow[dr, dashed, "j"'] & & \widetilde{E}^*(X^3) \arrow[ll, "i"'] \arrow[dr, dashed, "j"'] & & \cdots \arrow[ll, "i"'] \\
& \widetilde{E}^*(X^1/X^0) \arrow[ur, "k"'] & & \widetilde{E}^*(X^2/X^1) \arrow[ur, "k"'] & & \widetilde{E}^*(X^3/X^2) \arrow[ur, "k"'] & & \cdots &
\end{tikzcd}$$

Each triangle is the long exact sequence of the pair (X^k, X^{k-1}) and the dashed arrows increase the degree of the cohomology group by one.

This diagram gives an exact triangle. Define $E = \oplus_p \widetilde{E}^*(X^p/X^{p-1})$ and $F = \oplus_p \widetilde{E}^*(X^p)$. Then we can rewrite the above diagram as

$$\begin{tikzcd}
F \arrow[dr, dashed, "j"'] & & F \arrow[ll, "i"'] \\
& E. \arrow[ur, "k"'] &
\end{tikzcd}$$

We have a grading on E and F given by $E^{p,q} = \widetilde{E}^{p+q}(X^p/X^{p-1})$ and $F^{p,q} = \widetilde{E}^{p+q}(X^p)$. Then i, j and k have bidegree $(-1, 1), (1, 0)$ and $(0, 0)$ respectively.

It remains to see that this gives us a spectral sequence. Let $(F_r, E_r, i_r, j_r, k_r)$ be the $(r-1)$st derived couple of the above exact couple. Note that the grading of E and F carry over to E_r and F_r. To show that $E_r^{p,q}$ and $d_r = j_r k_r$ form a spectral sequence, the only thing to check is that the bidegree of d_r is $(r, -(r-1))$. This is purely book keeping[†].

6.1.2 The E_∞ page

Definition 6.5. Given a spectral sequence $(E_r^{*,*}, d_r)$, suppose there exists a function $r(p,q) \in \mathbb{N}_{>0}$ such that
$$E_r^{p,q} \cong E_{r+1}^{p,q},$$
for all $r \geq r(p,q)$. Then $(E_r^{*,*}, d_r)$ is said to *stabilise* and, in this case, the *limit page* E_∞ is defined by
$$E_\infty^{p,q} = \operatorname*{colim}_r E_r^{p,q}.$$

Note that the limit page is defined more generally, not just when stable. But this definition will suffice for our purposes.

In practise we can often assume $E_r^{p,q} = 0$ for some fixed r and all $p, q < 0$. So all the non-zero modules are located in the first quadrant, when we visualise the E_r page as an integral lattice. Then, for fixed p, q and large enough r, the differentials going in and out of $E_r^{p,q}$ will be zero. At this point, passing to the next page will not change the modules: $E_{r+1}^{p,q} = E_r^{p,q}$. So the groups stabilise and $E_\infty^{p,q} = E_r^{p,q}$.

There is another simple condition that guarantees stability. If there are only finitely many non-zero modules in each E column, then d_r, which goes downward $r-1$ rows, is again zero for sufficiently large r. Analogously, stability is guaranteed if there are only finitely many non-zero modules in each E row. We will see that this ensures that the Atiyah Hirzebruch spectral sequence stabilises.

6.1.3 Convergence

We remind the reader that we are only concerned with cohomology. As such we will focus on decreasing filtrations and convergence with respect to decreasing filtrations.

Definition 6.6. A *(decreasing) filtration* $\mathcal{F}^*$ on an R-module A is a sequence of sub-modules
$$\{0\} \subset ... \subset \mathcal{F}^{n+1}A \subset \mathcal{F}^n A \subset \mathcal{F}^{n-1}A \subset ... \subset A.$$

Suppose H^* is a graded R-module. The motivating example is any cohomology theory. If we have a filtration $\mathcal{F}$ on H^* then $\mathcal{F}^n H^m = \mathcal{F}^n H^* \cap H^m$ is a filtration on H^m and
$$\mathcal{F}^p H^{p+q} \big/ \mathcal{F}^{p+1} H^{p+q}$$
forms a bigraded module.

[†]We prove this by induction. The differential $d_1 = jk$ has bidegree $(1,0)$ since j has bidegree $(1,0)$ and k has bidegree $(0,0)$.

In the step case, assume that i_r, j_r and k_r have bidegree $(-1,1), (r, -(r-1))$ and $(0,0)$ respectively. Since $i_{r+1} = i_r|_{i_r(F_r)}$, the bidegree of i_{r+1} is equal to the bidegree of i_r. Similarly, the bidegree of k_{r+1} is equal to the bidegree of k_r, as $k_{r+1}[e] = k_r(e)$. Recall that j_{r+1} is defined by $j^{r+1}(i^r(f)) = [j^r f]$. If $i^r(f) \in F_r^{p,q}$, then $f \in F_r^{p+1,q-1}$ and consequently, $j^r(f) \in E_r^{p+r+1,q-r}$. Thus, j^{r+1} has bidegree $(r+1, -r)$. Therefore, the degree of d_{r+1} is $(r+1, -r)$ as required.

Definition 6.7. A spectral sequence $(E_r^{*,*}, d_r)$ *converges* to a graded R-module H^* if

$$E_\infty^{p,q} \cong \mathcal{F}^p H^{p+q} / \mathcal{F}^{p+1} H^{p+q},$$

for some filtration $\mathcal{F}$ of H^*. In this case, we write

$$E_1^{p,q} \Rightarrow H^{p+q} \text{ or } E_2^{p,q} \Rightarrow H^{p+q}.$$

Maps of spectral sequences

Definition 6.8. Let $(E_r^{*,*}, d_r)$ and $(\hat{E}_r^{*,*}, \hat{d}_r)$ be two spectral sequences. A *morphism of spectral sequences* is a sequence of homomorphisms $f_r : E_r^{*,*} \to \hat{E}_r^{*,*}$ of bigraded modules, of bidegree $(0,0)$, such that

1. Each f_r commutes with the corresponding differentials: $f_r \circ d_r = \hat{d}_r \circ f_r$.

2. Each f_{r+1} is the induced map of f_r on homology: that is, f_{r+1} is the composite

$$E_{r+1}^{*,*} \xrightarrow{\cong} H(E_r^{*,*}, d_r) \xrightarrow{H(f_r)} H(\hat{E}_r^{*,*}, \hat{d}_r) \xrightarrow{\cong} \hat{E}_{r+1}^{*,*}.$$

Since we often do not care about the first few pages of a spectral sequence, we allow the sequences of homomorphisms f_r to start at $r = 2$ or even later.

If both $E_r^{*,*}$ and $\hat{E}_r^{*,*}$ stabilise, then we get a map $f_\infty : E_\infty^{*,*} \to \hat{E}_\infty^{*,*}$ by taking f_r for large r. It is not too difficult to see that if $f_n : E_n \to \hat{E}_n$ is an isomorphism, then f_r is also, for $n \leq r \leq \infty$.

Suppose that $E_r^{*,*}$ and $\hat{E}_r^{*,*}$ are both built from exact couples (E, F, i, j, k) and $(\hat{E}, \hat{F}, \hat{\imath}, \hat{\jmath}, \hat{k})$ respectively. Suppose $f : E \to \hat{E}$ commutes with the first differential:

$$fjk = \hat{\jmath}\hat{k}f.$$

Then f induces a map of spectral sequences $\{f_r\}$ by defining $f_1 = f$ and f_{r+1} the map induced by f_r on homology. (We omit the proof that f_r commutes with the r-th differential.)

6.2 The Atiyah Hirzebruch Spectral Sequence

This spectral sequence computes the cohomology of CW complexes, for any cohomology theory. It was first published by Atiyah and Hirzebruch in [4] but Adams states that they were probably invented by G. W. Whitehead in the 1950's [2, p. 214].

Suppose X is a CW complex with basepoint x_0. Let $\mathcal{C}_p$ be the set of p-cells of X. Define the corresponding reduced version

$$\widetilde{\mathcal{C}}_p = \begin{cases} \mathcal{C}_p & \text{if } p \neq 0, \\ \mathcal{C}_p - \{x_0\} & \text{if } p = 0. \end{cases}$$

Then the p-th cellular chain complex $C_p(X)$ is the free abelian group with basis $\mathcal{C}_p$. Similarly, the reduced p-th cellular chain complex $\widetilde{C}_p(X)$ is the free abelian group with basis $\widetilde{\mathcal{C}}_p$. Define the reduced and unreduced cellular cochain complexes with coefficients in G

$$C^*(X;G) = \mathrm{Hom}_{\mathbb{Z}}(C_*(X), G) \cong \prod_{\mathcal{C}_p} G \quad \text{and} \quad \widetilde{C}^*(X;G) = \mathrm{Hom}_{\mathbb{Z}}(\widetilde{C}_*(X), G) \cong \prod_{\widetilde{\mathcal{C}}_p} G.$$

Theorem 6.9. *Let e be an unreduced cohomology theory and X a finite-dimensional CW complex. Then there exist spectral sequences $(E_r^{*,*}, d_r)$ and $(\hat{E}_r^{*,*}, \hat{d}_r)$ with*

$$E_1^{p,q} = e^{p+q}(X^p/X^{p-1}) \cong C^p\left(X; e^q(*)\right), \quad E_2^{p,q} \cong H^p\left(X, e^q(*)\right) \Rightarrow e^{p+q}(X),$$

and

$$\hat{E}_1^{p,q} = \widetilde{e}^{p+q}(X^p/X^{p-1}) \cong \widetilde{C}^p\left(X; e^q(*)\right), \quad \hat{E}_2^{p,q} \cong \widetilde{H}^p\left(X, e^q(*)\right) \Rightarrow \widetilde{e}^{p+q}(X).$$

Proof. For our purposes, we will only need the second spectral sequence. Therefore, we will only prove the existence of this spectral sequence. The other one follows in a similar manner.

Suppose X is a finite dimensional CW complex and let X^p be the p-skeleton of X with inclusion $X^p \hookrightarrow X$. Define $X^p = \{x_0\}$ for $p < 0$, where x_0 is the basepoint of X. Let d be the dimension of X.

We build the spectral sequence $\hat{E}_r^{p,q}$ using the exact couple in subsection 6.1.1, using the skeletal filtration of X. Specifically, define $\hat{E}_1^{p,q} = \widetilde{e}^{p+q}(X^p/X^{p-1})$ and $\hat{F}_r^{p,q} = \widetilde{e}^{p+q}(X^p)$. Let i, j, k be the maps of this exact couple, following the conventions of subsection 6.1.1.

We have that $\widetilde{e}^{p+q}(S^p) \cong e^q(*)^\dagger$. Consequently,

$$\hat{E}_1^{p,q} \cong \widetilde{e}^{p+q}(\vee_{\widetilde{\mathcal{C}}_p} S^p) \cong \prod_{\widetilde{\mathcal{C}}_p} \widetilde{e}^{p+q}(S^p) \cong \prod_{\widetilde{\mathcal{C}}_p} e^q(*) \cong \widetilde{C}^p\left(X; e^q(*)\right).$$

Notice that the diagram

$$\begin{tikzcd}
\hat{E}_1^{p,q} \arrow[r, "\cong"] \arrow[d, "jk=\hat{d}_1"'] & \widetilde{C}^p\left(X; e^q(*)\right) \arrow[d, "\partial"] \\
\hat{E}_1^{p+1,q} \arrow[r, "\cong"] & \widetilde{C}^{p+1}\left(X; e^q(*)\right)
\end{tikzcd}$$

commutes, where ∂ is the cellular coboundary map. This implies $\hat{E}_2^{p,q} \cong \widetilde{H}^p\left(X, e^q(*)\right)$, as required.

We know that $\widetilde{H}^p\left(X, e^q(*)\right)$ is zero for $p < 0$ and $p > d$. Thus, the spectral sequences stabilises. All that remains is to show that the spectral sequence converges to $\widetilde{e}^{p+q}(X)$.

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† This can be proven directly from the axioms of cohomology, or through Browns representability theorem: Let A be a spectrum representing $\widetilde{e}$. Then we can compute

$$\widetilde{e}^{p+q}(S^p) \cong \operatorname*{colim}_{n\to\infty}[\Sigma^n S^p, A_{n+p+q}] \cong \operatorname*{colim}_{n\to\infty}[\Sigma^n S^0, A_{n+q}] \cong \widetilde{e}^q(S^0) \cong e^q(*).$$

Define a (decreasing) filtration of $\widetilde{e}^n(X)$

$$\widetilde{e}^n(X) = \mathcal{F}^0\widetilde{e}^n(X) \supset \mathcal{F}^1\widetilde{e}^n(X) \supset ... \supset \mathcal{F}^{d+1}\widetilde{e}^n(X) = 0$$

where $\mathcal{F}^p\widetilde{e}^n(X) = \text{Ker}\left(\widetilde{e}^n(X) \to \widetilde{e}^n(X^{p-1})\right)$ for $p > 0$ and $\mathcal{F}^0\widetilde{e}^n(X) = \widetilde{e}^n(X)$. We claim that $\hat{E}_\infty^{p,q} \cong \mathcal{F}^p\widetilde{e}^{p+q}(X)/\mathcal{F}^{p+1}\widetilde{e}^{p+q}(X)$.

Consider the long exact sequence for the $(r-1)$st derived couple

$$\hat{E}_r^{p-r+1,q+r-2} \xrightarrow{k_r} \hat{F}_r^{p-r+1,q+r-2} \xrightarrow{i_r} \hat{F}_r^{p-r,q+r-1} \xrightarrow{j_r} \hat{E}_r^{p,q} \xrightarrow{k_r} \hat{F}_r^{p,q} \xrightarrow{i_r} \hat{F}_r^{p-1,q+1} \xrightarrow{j_r} \hat{E}_r^{p+r-1,q-r+2}.$$

Since $\hat{F}_r^{p-r,q+r-1} \subset \hat{F}_{r-1}^{p-r-1,q+r} \subset ... \subset \hat{F}_1^{p-2r+1,q+2r-1} = \widetilde{e}^{p+q}(X^{p-2r+1})$, the second $\hat{F}$ term in the above long exact sequence is 0, for large enough r. This implies $\hat{E}_r^{p,q}$ is isomorphic to the kernel of $\hat{F}_r^{p,q} \to \hat{F}_r^{p-1,q+1}$.

But $\hat{F}_r^{p,q} = i_{r-1}(\hat{F}_{r-1}^{p+1,q-1})$, so all the elements of $\hat{F}_r^{p,q}$ come from $\hat{F}_1^{p+r-1,q-r+1} = \widetilde{e}^{p+q}(X^{p+r-1})$. If r is large enough, $\widetilde{e}^{p+q}(X^{p+r-1}) = \widetilde{e}^{p+q}(X)$. Similarly, all the elements of $\hat{F}_r^{p-1,q+1}$ come from $\widetilde{e}^{p+q}(X)$ for large enough r. Thus, $\hat{E}_\infty^{p,q}$ is isomorphic to the quotient of Ker $\left(\widetilde{e}^{p+q}(X) \to \hat{F}_r^{p-1,q+1}\right)$ by Ker $\left(\widetilde{e}^{p+q}(X) \to \hat{F}_r^{p,q}\right)$. That is,

$$\hat{E}_\infty^{p,q} \cong \mathcal{F}^p\widetilde{e}^{p+q}(X)/\mathcal{F}^{p+1}\widetilde{e}^{p+q}(X),$$

as required. $\square$

This spectral sequence also works for infinite dimensional X. The proof is mostly the same, but it's necessary to take limits at some points. See [2, part III, §7]. However, the finite case will be satisfactory for our purposes.

Chapter 7

K-Theory of Stunted Projective Spaces

In this chapter, we will compute the K-theory of stunted projective space and their corresponding Adams' operations. This will form a major component of the proof that $\mathbb{R}\mathrm{P}^{n+\rho n}/\mathbb{R}\mathrm{P}^{n-1}$ is not coreducible. However, we will include every stunted projective space, for completeness, even if it is not used in the vector fields problem. We will generally use the notation $\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m$ instead of $\mathbb{R}\mathrm{P}^n_{m+1}$ to emphasise that we only consider actual stunted projective space, as opposed to the general definition $\mathbb{R}\mathrm{P}^n_m = \mathrm{Th}(\mathbb{R}\mathrm{P}^{n-m}, m\xi)$.

We will proceed by examining

1. the complex K-theory of $\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^m$,
2. the complex K-theory of $\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m$,
3. the real K-theory of $\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m$,

for $n > m \geq 0$.

This chapter follows section 7 of [1] closely.

Cohomology of stunted projective spaces

In this chapter, we will use the Atiyah Hirzebruch spectral sequence extensively in the computations of $\widetilde{K}_{\mathbb{F}}(\mathbb{F}\mathrm{P}^n/\mathbb{F}\mathrm{P}^m)$. It is therefore convenient to state the relevant cohomology groups which form the E_2 pages. When $m = 0$, these are given by [9, Theorem 3.19]:

$$\widetilde{H}^k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = n \text{ odd}, \\ \mathbb{Z}_2 & \text{if } k \text{ even and } 0 < k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

$$\widetilde{H}^k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & \text{if } 0 < k \le n, \\ 0 & \text{otherwise.} \end{cases}$$

$$\widetilde{H}^k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k \text{ even and } 0 < k \le 2n, \\ 0 & \text{otherwise.} \end{cases}$$

These groups can be computed by examining the cellular chains of $\mathbb{F}\mathrm{P}^n$. In the real case, there is one cell in every dimension $0 \le m \le n$ and in the complex case, one cell in every even dimension $0 \le m \le 2n$. For $n \ge m \ge 0$, pruning the first m cells in the cellular chain of $\mathbb{F}\mathrm{P}^n$ produces the cellular chain of $\mathbb{F}\mathrm{P}^n/\mathbb{F}\mathrm{P}^m$. The attaching maps of the higher-dimension cells remains unaffected, with the exception of the $(m+1)$-cell in the real case and the $(2m+2)$-cell in the complex case, which now have trivial attaching map. Therefore, we compute the cohomology of $\mathbb{F}\mathrm{P}^n/\mathbb{F}\mathrm{P}^m$ in the same way as for $\mathbb{F}\mathrm{P}^n$. For simplicity in writing the cohomology groups, we switch back to the notation $\mathbb{F}\mathrm{P}^n_m = \mathbb{F}\mathrm{P}^n/\mathbb{F}\mathrm{P}^{m-1}$.

$$\widetilde{H}^k(\mathbb{R}\mathrm{P}^n_m;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = n \text{ odd or } k = m \text{ even}, \\ \mathbb{Z}_2 & \text{if } k \text{ even and } m < k \le n, \\ 0 & \text{otherwise.} \end{cases} \tag{7.1}$$

$$\widetilde{H}^k(\mathbb{R}\mathrm{P}^n_m;\mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & \text{if } m \le k \le n, \\ 0 & \text{otherwise.} \end{cases} \tag{7.2}$$

$$\widetilde{H}^k(\mathbb{C}\mathrm{P}^n_m;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k \text{ even and } 2m \le k \le 2n, \\ 0 & \text{otherwise.} \end{cases}$$

The Standard Projection

Definition 7.1. Writing

$$\mathbb{R}\mathrm{P}^{2n+1} = S^{2n+1}/(x \sim -x) \text{ and } \mathbb{C}\mathrm{P}^n = S^{2n+1}/(x \sim \lambda x \text{ for } \lambda \in S^1),$$

define the *standard projection* $\pi : \mathbb{R}\mathrm{P}^{2n+1} \to \mathbb{C}\mathrm{P}^n$ by $\pi([x]) = [x]$.

This is well defined since the equivalence classes $[x] \in \mathbb{R}\mathrm{P}^{2n+1}$ are finer than the equivalence classes $[x] \in \mathbb{C}\mathrm{P}^n$†.

Note that $\pi : \mathbb{R}\mathrm{P}^{2n-1} \to \mathbb{C}\mathrm{P}^{n-1}$ is induced by the map $\mathbb{R}^{2n} \to \mathbb{C}^n$ that sends $(x_1, x_2, ..., x_{2n})$ to $(x_1 + ix_2, ..., x_{2n-1} + ix_{2n})$.

Using this definition, we see that the image of the subcomplex $\mathbb{R}\mathrm{P}^{2m-1} \subset \mathbb{R}\mathrm{P}^{2n-1}$ is contained in $\mathbb{C}\mathrm{P}^{m-1} \subset \mathbb{C}\mathrm{P}^{n-1}$.

†That is, for every $x \in S^{2n+1}$, the equivalence class $[x] \in \mathbb{C}\mathrm{P}^n$ contains the equivalence class $[x] \in \mathbb{R}\mathrm{P}^{2n+1}$.

Notation

Let ξ and η be the canonical line bundles over $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ respectively. If we project $\mathbb{R}\mathrm{P}^n$ onto $\mathbb{R}\mathrm{P}^{n-1}$, then ξ over $\mathbb{R}\mathrm{P}^n$ restricts to ξ over $\mathbb{R}\mathrm{P}^{n-1}$. Similarly for η. This justifies not displaying the index n in our notation. We introduce the following elements λ, μ, ν:

$$\begin{aligned}
\lambda &= \xi - \epsilon^1 \in \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n),\\
\mu &= \eta - \epsilon^1 \in \widetilde{K}_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n),\\
\nu &= c\lambda \in \widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n),
\end{aligned}$$

where ϵ^1 is the trivial line bundle. These bundles will play a large role in describing the K-theory of stunted projective spaces. By the same reasoning as above, we need not display the index n in the notation for λ, μ and ν.

For odd $n = 2k-1$, we have $\nu = \pi^*\mu$ for $\mu \in \widetilde{K}_{\mathbb{C}}(\mathbb{C}\mathrm{P}^{k-1})$ by the following lemma:

Lemma 7.2. *Let ξ be the canonical real line-bundle over $\mathbb{R}\mathrm{P}^{2n-1}$. Let η be the canonical line-bundle over $\mathbb{C}\mathrm{P}^{n-1}$. Then $c\xi \cong \pi^*\eta$, where c is complexification.*

We delay the proof of this lemma, until after we have introduced Stiefel-Whitney and Chern classes.

Stiefel-Whitney and Chern Classes

Characteristic classes are ways of associating algebraic invariants, specifically cohomology classes, to vector bundles. Stiefel-Whitney and Chern classes are two such characteristic classes, defined for real and complex vector bundles respectively.

All results in this section are well known and therefore we shall omit their proofs. These can all be found in [10, §3.1].

Recall that $\mathrm{Vect}_{\mathbb{F}}(X)$ is the isomorphism classes of $\mathbb{F}$-vector bundles over X. We present the Stiefel-Whitney classes first:

Theorem 7.3. *There exist functions $w_i : \mathrm{Vect}_{\mathbb{R}}(X) \to H^i(X;\mathbb{Z}_2)$ such that, for all real bundles $E \to X$,*

1. *w_i is natural: $w_i(f^*(E)) = f^*(w_i(E))$, where $f^*(E)$ is a pullback of E, and $f^*(w_i(E))$ is the induced map on cohomology;*
2. *$w(E_1 \oplus E_2) = w(E_1) \smile w(E_2)$, where $w(E) := 1 + w_1(E) + w_2(E) + ... \in H^*(X;\mathbb{Z}_2)$;*
3. *$w_i(E) = 0$ if i is greater than the dimension of E†;*
4. *the class $w_1(\xi)$ is the generator of $H^1(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2) \cong \mathbb{Z}_2$, where ξ is the canonical line bundle on $\mathbb{R}\mathrm{P}^\infty$.*

†This ensures that the sum $w(E) := 1 + w_1(E) + w_2(E) + ...$ is always finite.

The cohomology class $w_i(E)$ is called the *i-th Stiefel-Whitney class* of E and $w(E)$ is called the *total Stiefel-Whitney class* of E.

We have a similar theorem for Chern classes:

Theorem 7.4. *There exist functions* $c_i : \mathrm{Vect}_{\mathbb{C}}(X) \to H^{2i}(X;\mathbb{Z})$ *such that, for all complex bundles* $E \to X$,

1. c_i *is natural:* $c_i(f^*(E)) = f^*(c_i(E))$, *where* $f^*(E)$ *is a pullback of* E, *and* $f^*(c_i(E))$ *is the induced map on cohomology;*

2. $c(E_1 \oplus E_2) = c(E_1) \smile c(E_2)$, *where* $c(E) := 1 + c_1(E) + c_2(E) + ... \in H^*(X;\mathbb{Z})$*;*

3. $c_i(E) = 0$ *if* i *is greater than the dimension of* E*;*

4. *the class* $c_1(\eta)$ *is a generator of* $H^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \cong \mathbb{Z}$, *specified in advance, where* η *is the canonical line bundle on* $\mathbb{C}\mathrm{P}^\infty$.

The cohomology class $c_i(E)$ is called the *i-th Chern class* of E and $c(E)$ is called the *total Chern class* of E;

It turns out that, in most cases, the first Stiefel-Whitney classes classify the real line bundles and the first Chern classes classify the complex line bundles: Recall that the isomorphism classes of one-dimensional vector bundles $\mathrm{Vect}^1_{\mathbb{F}}(X)$ form a group with respect to the tensor product.

Theorem 7.5. *The first class* $w_1 : \mathrm{Vect}^1_{\mathbb{R}}(X) \to H^1(X;\mathbb{Z}_2)$ *is a homomorphism, and an isomorphism if* X *is homotopy equivalent to a CW complex. The same applies for* $c_1 : \mathrm{Vect}^1_{\mathbb{C}}(X) \to H^2(X;\mathbb{Z})$.

Proof of Lemma 7.2. If $n = 1$, then $H^2(\mathbb{R}\mathrm{P}^{2n-1};\mathbb{Z}) = 0$. But the complex line bundles are classified by their Chern class c_1 in $H^2(\mathbb{R}\mathrm{P}^{2n-1};\mathbb{Z})$. Since both $c\xi$ and $\pi^*\eta$ are complex line bundles, they must be isomorphic.

Suppose $n > 1$. Now, $H^2(\mathbb{R}\mathrm{P}^{2n-1};\mathbb{Z}) \cong \mathbb{Z}_2$. We will show that the Chern classes of $c\xi$ and $\pi^*\eta$ are non-trivial. Then we must have $c_1 c\xi = 1 = c_1\pi^*\eta$ and so $c\xi \cong \pi^*\eta$.

To show $c_1 c\xi$ is non-trivial, it suffices to show that $c\xi$ is non-trivial. We know that $w(\xi) = 1 + x$, where x is the generator of $H^1(\mathbb{R}\mathrm{P}^{2n-1};\mathbb{Z}_2)$. Let r be the 'realification' map, defined in subsection 1.3.2. Then $w(rc\xi) = w(\xi \oplus \xi) = 1 + x^2$. It follows that $c\xi$ is non-trivial.

To show $c_1\pi^*\eta \neq 0$, note first that $c_1\pi^*\eta = \pi^*c_1\eta$. Since η is the canonical line bundle, $c_1\eta$ is a generator of $\mathbb{Z} \cong H^2(\mathbb{C}\mathrm{P}^{n-1};\mathbb{Z})$ (c.f. [10, Theorem 3.2]). But a generator is sent to 1 by the map

$$\mathbb{Z} \cong H^2(\mathbb{C}\mathrm{P}^{n-1};\mathbb{Z}) \xrightarrow{\pi^*} H^2(\mathbb{R}\mathrm{P}^{2n-1};\mathbb{Z}) \cong \mathbb{Z}_2,$$

and so $\pi^*c_1\eta = 1$. $\square$

7.1 The Complex K-Theory of $\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^m$

Theorem 7.6. *$K_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n) \cong \mathbb{Z}[\mu]/\langle \mu^{n+1}\rangle$. The projection $q : \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^m$ induces an isomorphism $\widetilde{K}_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^m) \xrightarrow{\cong} \langle \mu^{m+1}\rangle \subset \widetilde{K}_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n)$. (That is, $\widetilde{K}_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^m)$ is isomorphic to the ideal of $K_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n)$ generated by μ^{m+1}). The Adams operations on $K_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n)$ are given by*

$$\Psi^k_{\mathbb{C}}(\mu^s) = \left((1+\mu)^k - 1\right)^s.$$

When k is negative, we interpret the power $(1+\mu)^k$ by the binomial expansion

$$(1+\mu)^k = 1 + k\mu + \frac{k(k-1)}{2!}\mu^2 + \dots$$

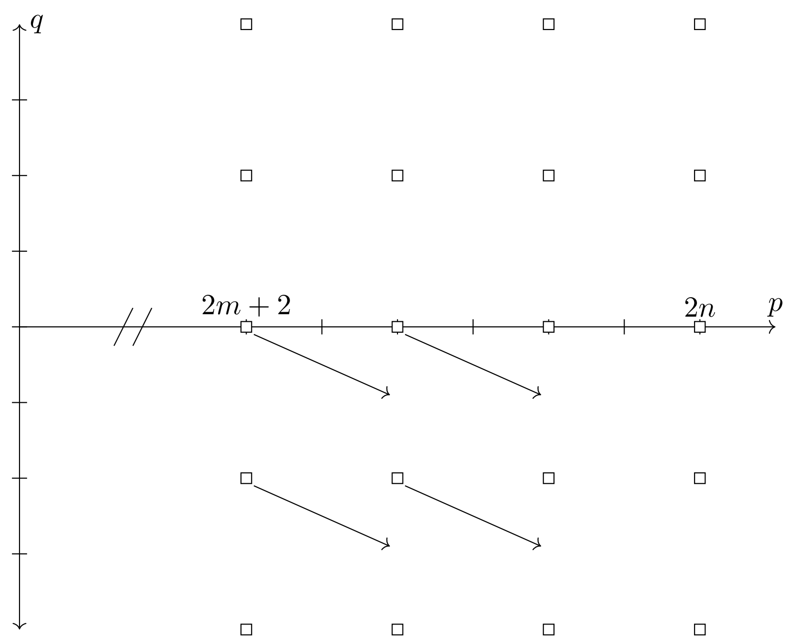
which is finite since $\mu^{n+1} = 0$.

Since Adams operations are natural, their behaviour on $\widetilde{K}_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^m)$ is determined by $\Psi^k_{\mathbb{C}} : K_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n) \to K_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n)$ and the isomorphism $\widetilde{K}_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^m) \xrightarrow{\cong} \langle \mu^{m+1}\rangle \subset \mathbb{Z}[\mu]/\langle \mu^{n+1}\rangle$.

Proof. The first statement is a well known result (c.f. [10, Proposition 2.24]), so we will omit its proof. However, we will take the opportunity to illustrate the power of the Atiyah Hirzebruch spectral sequence, by computing the group structure of $\widetilde{K}_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^m)$. Note that the spectral sequence $(E_r^{*,*}, d_r)$ for $\widetilde{K}_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^m)$ stabilises on the E_2 page. Why? The E_2 page is given by

$$E_2^{p,q} \cong H^p\left(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^m; K^q_{\mathbb{C}}(*)\right) \cong \begin{cases} \mathbb{Z} & \text{if } q \text{ even and } 2m+2 \le p \le 2n \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

So the E_2 page looks like, where the squares are $\mathbb{Z}$:



As usual, we have omitted many of the differentials. The total degree of every non-zero term is always even. But the total degree of every differential d_r is odd. Thus, every differential has a zero term as their source or target. Then

$$\widetilde{K}^s_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^m) \cong E_\infty^{p,s-p} = E_2^{p,s-p} = 0$$

if s is odd. If s is even then $\widetilde{K}^s_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^m) \cong E_2^{p,s-p}$ has a filtration such that the successive quotients are $n-m$ copies of $\mathbb{Z}$. It follows that $\widetilde{K}^s_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^m)$ is free abelian on $n-m$ generators, for s even. This determines the group structure of $\widetilde{K}_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^m)$. The generators and ring structure of $\widetilde{K}_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n)$ can be determined using the Chern classes. See [10, Proposition 2.24] or [1, Theorem 7.2].

To prove the second statement, examine the long exact sequence of the pair $(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^m)$:

$$0 \to \widetilde{K}_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^m) \xrightarrow{q^*} \widetilde{K}_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n) \xrightarrow{i^*} \widetilde{K}_{\mathbb{C}}(\mathbb{C}\mathrm{P}^m) \xrightarrow{\delta} \ldots$$

So $\widetilde{K}_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^m)$ is isomorphic to the kernel of i^*. But the kernel of i^* is the subgroup of $\widetilde{K}_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n)$ generated by $\mu^{m+1}, \mu^{m+2}, \ldots, \mu^n$. The second statement now follows.

It remains to compute $\Psi^k_{\mathbb{C}}(\mu^s)$. By Theorem 1.47, we have $\Psi^k_{\mathbb{C}}(\eta) = \eta^k$; that is, $\Psi^k_{\mathbb{C}}(1+\mu) = (1+\mu)^{k}$†. It follows that $\Psi^k_{\mathbb{C}}(\mu) = (1+\mu)^k - 1$ and consequently

$$\Psi^k_{\mathbb{C}}(\mu^s) = \Big((1+\mu)^k - 1\Big)^s,$$

as required. □

Define $\mu^{(m+1)}$ to be the element in $\widetilde{K}_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^m)$ that is sent to μ^{m+1} by q^* from Theorem 7.6. The following lemma justifies not displaying n in the notation.

Lemma 7.7. *The map induced by the inclusion $i : \mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^{n+1}/\mathbb{C}\mathrm{P}^m$ sends $\mu^{(m+1)} \in \widetilde{K}_{\mathbb{C}}(\mathbb{C}\mathrm{P}^{n+1}/\mathbb{C}\mathrm{P}^m)$ to the corresponding $\mu^{(m+1)} \in \widetilde{K}_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^m)$. Similarly for the projection $q : \mathbb{C}\mathrm{P}^{n+1}/\mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^m$.*

Proof. If you take the canonical line bundle on $\mathbb{C}\mathrm{P}^{n+1}$ and restrict to $\mathbb{C}\mathrm{P}^n$, the result is again the canonical line bundle. It follows that $i^*_{\mathbb{C}} : K(\mathbb{C}\mathrm{P}^{n+1}) \to K(\mathbb{C}\mathrm{P}^n)$ sends μ^{m+1} to μ^{m+1}. Then the result follows from commutativity of the square

$$\begin{tikzcd}
\widetilde{K}(\mathbb{C}\mathrm{P}^{n+1}/\mathbb{C}\mathrm{P}^m) \arrow[rrr, "i^*_{\mathbb{C}}"] \arrow[ddd, "q^*_{\mathbb{C}}"'] \arrow[dr, phantom, "\ni"] & & & \widetilde{K}(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^m) \arrow[ddd, "q^*_{\mathbb{C}}"] \arrow[dl, phantom, "\in"] \\
& \mu^{(m+1)} \arrow[d, mapsto] & \mu^{(m+1)} \arrow[d, mapsto] & \\
& \mu^{m+1} \arrow[r, mapsto] \arrow[dl, phantom, "\in"] & \mu^{m+1} \arrow[dr, phantom, "\in"] & \\
K(\mathbb{C}\mathrm{P}^{n+1}) \arrow[rrr, "i^*_{\mathbb{C}}"] & & & K(\mathbb{C}\mathrm{P}^n).
\end{tikzcd}$$

The second statement follows in the same manner. □

† Note that the multiplicative identity in $K_{\mathbb{F}}(X)$ is the trivial line bundle ϵ^1.

7.2 The Complex K-Theory of $\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m$

The complex K-theory of $\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m$ is more difficult. We need to consider two cases, $m = 2t$ and $m = 2t+1$.

Consider the diagram

$$\begin{tikzcd}
\mathbb{R}\mathrm{P}^{2n+1} \arrow[r, two heads, "\pi"] \arrow[d, two heads] & \mathbb{C}\mathrm{P}^n \arrow[d, two heads, "q"] \\
\mathbb{R}\mathrm{P}^{2n+1}/\mathbb{R}\mathrm{P}^{2t} \arrow[r, dashed, "\omega_1"] & \mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^t.
\end{tikzcd}$$

We know that the image of $\mathbb{R}\mathrm{P}^{2t} \subset \mathbb{R}\mathrm{P}^{2n+1}$ under π is contained in $\mathbb{C}\mathrm{P}^t \subset \mathbb{C}\mathrm{P}^n$. So $q \circ \pi$ sends $\mathbb{R}\mathrm{P}^{2t}$ to the basepoint. Thus, the quotient map ω_1 exists. In an analogous way, the standard projection π also factors to give $\omega_2 : \mathbb{R}\mathrm{P}^{2n+1}/\mathbb{R}\mathrm{P}^{2t+1} \to \mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^t$.

Define $\nu_1^{(t+1)} = \omega_1^*\mu^{(t+1)}$ and $\nu_2^{(t+1)} = \omega_2^*\mu^{(t+1)}$. Note that $\nu_1^{(1)} = \nu$, by Lemma 7.2. The following lemma explains the choice of notation.

Lemma 7.8. *We have*

$$\nu_2^{(t+1)} = i^*\nu_1^{(t+1)} \text{ and } q_1^*\nu_1^{(t+1)} = q_2^*\nu_2^{(t+1)} = \nu^{t+1},$$

where $i : \mathbb{R}\mathrm{P}^{2n+1}/\mathbb{R}\mathrm{P}^{2t+1} \to \mathbb{R}\mathrm{P}^{2n+1}/\mathbb{R}\mathrm{P}^{2t}$ *is the inclusion and* $q_1 : \mathbb{R}\mathrm{P}^{2n+1} \to \mathbb{R}\mathrm{P}^{2n+1}/\mathbb{R}\mathrm{P}^{2t}$, $q_2 : \mathbb{R}\mathrm{P}^{2n+1} \to \mathbb{R}\mathrm{P}^{2n+1}/\mathbb{R}\mathrm{P}^{2t+1}$ *are the quotient maps. Moreover,*

$$\nu_1^{(t+1)} = q_3^*\nu_2^{(t+1)},$$

where $q_3 : \mathbb{R}\mathrm{P}^{2n+1}/\mathbb{R}\mathrm{P}^{2t} \to \mathbb{R}\mathrm{P}^{2n+1}/\mathbb{R}\mathrm{P}^{2t+1}$ *is the quotient map.*

Proof. The diagram

$$\begin{tikzcd}
\mathbb{R}\mathrm{P}^{2n+1} \arrow[r, "\pi"] \arrow[d, two heads, "q_2"] \arrow[dd, two heads, bend right=60, "q_1"'] & \mathbb{C}\mathrm{P}^n \arrow[d, two heads, "\tilde{q}"] \\
\mathbb{R}\mathrm{P}^{2n+1}/\mathbb{R}\mathrm{P}^{2t+1} \arrow[r, "\omega_2"] \arrow[d, "i"] & \mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^t \\
\mathbb{R}\mathrm{P}^{2n+1}/\mathbb{R}\mathrm{P}^{2t} \arrow[ur, "\omega_1"'] &
\end{tikzcd}$$

commutes. Then the first two results follow from the induced diagram on K-theory: The first result is immediate and we obtain the second by observing

$$q_j^*\nu_j^{(t+1)} = q_j^*\omega_j^*\mu^{(t+1)} = \pi^*\tilde{q}^*\mu^{(t+1)} = \nu^{t+1},$$

for $j = 1, 2$.

The final statement follows from the fact that the triangle

$$\begin{tikzcd}
\mathbb{R}\mathrm{P}^{2n+1}/\mathbb{R}\mathrm{P}^{2t+1} \arrow[r, "\omega_2"] & \mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^t \\
\mathbb{R}\mathrm{P}^{2n+1}/\mathbb{R}\mathrm{P}^{2t} \arrow[u, "q_3"] \arrow[ur, "\omega_1"'] &
\end{tikzcd}$$

commutes. $\square$

There are statements analogous to Lemma 7.7 for $\nu_1^{(t+1)}$ and $\nu_2^{(t+1)}$ justifying why n does not appear in the notation.

We are now in a position to state the complex K theory of $\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m$ and the corresponding Adams operations.

Theorem 7.9. *1. Suppose $m = 2t$ is even. Then there is a group isomorphism $\widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{2t}) \cong \mathbb{Z}_{2^f}$ where $f = \lfloor \frac{1}{2}(n-m) \rfloor$.*

(a) The (non-unital) ring $\widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n)$ is generated by ν with two relations

$$\nu^2 = -2\nu \text{ and } \nu^{f+1} = 0.$$

(It follows that $2^f \nu = -2^{f-1}\nu^2 = \ldots = \nu^{f+1} = 0$.)

(b) For $m \neq 0$, the projection $q : \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{2t}$ induces an isomorphism from $\widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{2t})$ onto the ideal of $\widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n)$ generated by ν^{t+1}.

The Adams operations are given by

$$\Psi^k_{\mathbb{C}} \nu_1^{(t+1)} = \begin{cases} \nu_1^{(t+1)} & \text{if } k \text{ odd,} \\ 0 & \text{otherwise.} \end{cases}$$

2. Suppose $m = 2t+1$ is odd. Then there is an isomorphism (of non-unital rings)

$$\widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{2t+1}) \cong \mathbb{Z} \oplus \widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{2t+2}),$$

where the first summand is generated by $\nu_2^{(t+1)}$ and the second summand is embedded by the quotient $q : \mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{2t+1} \to \mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{2t+2}$.

The Adams operations are given by

$$\Psi^k_{\mathbb{C}} \nu_2^{(t+1)} = k^{t+1}\nu_2^{(t+1)} + \begin{cases} \frac{1}{2}k^{t+1}\nu_1^{(t+2)} & \text{if } k \text{ even,} \\ \frac{1}{2}(k^{t+1}-1)\nu_1^{(t+2)} & \text{otherwise.} \end{cases}$$

Since Adams operations $\Psi^k_{\mathbb{C}}$ are natural, the value of $\Psi^k_{\mathbb{C}}$ on the second summand of $\widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{2t+1}) \cong \mathbb{Z} \oplus \widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{2t+2})$ is given by part 1. of the above theorem.

We can write $\widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{2t})$ explicitly as

$$\nu_1^{(t+1)}\mathbb{Z}_{2^f}[\nu_1^{(t+1)}] \Bigg/ \left\langle \left(\nu_1^{(t+1)}\right)^2 - (-2)^{t+1}\nu_1^{(t+1)} \right\rangle.$$

Proof. We leave the results on the Adams operations to the end of the proof. We first examine the Atiyah Hirzebruch spectral sequence $(E_r^{*,*}, d_r)$ of $\widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$.

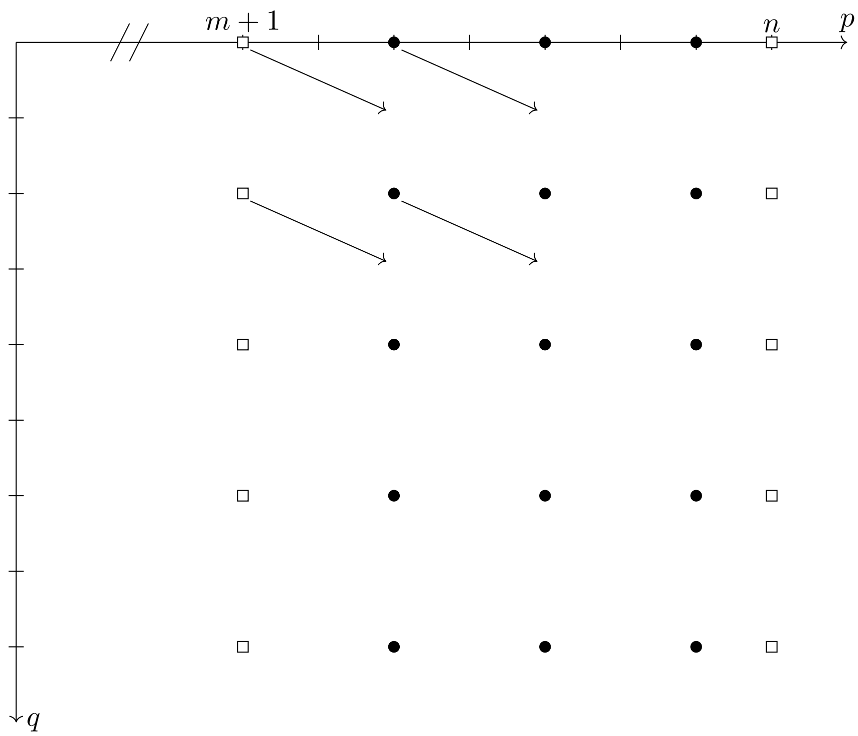
Recall that

$$K^q_{\mathbb{C}}(*) \cong \begin{cases} \mathbb{Z} & \text{if } q \text{ even,} \\ 0 & \text{otherwise,} \end{cases}$$

by Bott periodicity in chapter 1. Then by equation 7.1, the second page is given by

$$E_2^{p,q} \cong \widetilde{H}^p\left(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m, K^q_{\mathbb{C}}(*)\right) \cong \begin{cases} \mathbb{Z} & \text{if } q \text{ even and either } p = n \text{ odd or } p = m+1 \text{ even,} \\ \mathbb{Z}_2 & \text{if } m+1 < p \leq n \text{ even and } q \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

We will call the set of terms $\{E_r^{p,-p}\}_{p\in\mathbb{Z}}$ the main diagonal of the r-th page. We know that $\widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$ is built from the groups $E_\infty^{p,-p}$ along the main diagonal. Therefore, we often need only examine the fourth quadrant of each page, if the differentials from the other quadrants don't interact with the fourth quadrant. So this quadrant will be our main focus. For example, the fourth quadrant of the second page is drawn below, for n, m odd. As usual, we have omitted many of the differentials. The circles are $\mathbb{Z}_2$ and the squares are $\mathbb{Z}$.



Write $f = \lfloor \frac{1}{2}(n-2t) \rfloor$, where $m = 2t$ or $2t-1$, according to whether m is odd or even. (Note we have changed $m = 2t+1$ in the theorem statement to $m = 2t-1$.) The number of $\mathbb{Z}_2$ along the main diagonal of E_2 is f. These account for all the non-zero terms on the main diagonal, if m is even. If m is odd, then there is an additional $\mathbb{Z}$ term, at $E_2^{m+1,-m-1}$.

We want to show that $E_\infty^{p,-p} = E_2^{p,-p}$. Every non-zero term $E_2^{p,q}$, except when $p = n$ odd, has even total degree. But d_r has odd total degree. So the target or source of every differential is 0, except perhaps differentials with targets in $E_r^{p,q}$, for $p = n$ odd and q even.

By our proof of the Atiyah Hirzebruch spectral sequence, we know that $\widetilde{K}_{\mathbb{C}}(X)$ is filtered by the images of the groups $\widetilde{K}_{\mathbb{C}}(X/X^{p-1})$. So the elements

$$\nu_2^{(t+i)} \in \widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{2t+2i-1}),$$

for $i = 1, 2, ..., f$, yield generators for the filtration quotients. (We know they are generators since these filtration quotients are built from the $E_2^{p,-p}$ terms, for $p \neq m$ odd. So the filtration quotients are $\mathbb{Z}_2$ or 0. Thus, any non-trivial element is a generator.) Then, by passing to quotients, they yield generators for the E_2 terms $\mathbb{Z}_2$ along the main diagonal, apart from the $E_2^{m+1,-m-1}$ term when m is odd. Similarly, by passing to quotients, $\nu_2^{(t)}$ yields a generator for $\mathbb{Z} \cong E_2^{m+1,-m-1}$, when m odd.

Since these generators come from the filtration of $\widetilde{K}_{\mathbb{C}}(X)$, they must survive to the E_∞ page. Thus, the differentials out of these terms into $E_2^{n,q} \cong \mathbb{Z}$ (with n odd, q even) must be zero. Similarly, the differentials out of these terms into $E_r^{n,q}$ must be zero. Therefore, for all r, the differentials d_r with target or source on the main diagonal are zero and consequently, $E_\infty^{p,-p} = E_2^{p,-p}$.

We conclude that:

1. If $m = 2t$ is even, then $\widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$ has a filtration with successive quotients f copies of $\mathbb{Z}_2$, with generators the images of $\nu_2^{(t+1)}, ..., \nu_2^{(t+f)}$. So $\widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{2t})$ is embedded in $\widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n)$. Moreover, this embedding sends $\nu_1^{(t+1)}$ to ν^{t+1}. In the case $t = 0$, the generators are $\nu, \nu^2, ..., \nu^f$, so $\widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n)$, as a ring, is generated by ν.

2. If $m = 2t - 1$ is odd, then we have an additional $\mathbb{Z}$ term in the filtration of $\widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$. This term comes from $E_1^{m+1,-m-1} = \widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^{2t}/\mathbb{R}\mathrm{P}^{2t-1})$ and is generated by $\nu_2^{(t)}$. By examining the filtrations of $\widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{2t})$ and $\widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{2t-1})$, we see that

   $$0 \to \widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{2t}) \xrightarrow{q^*} \widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{2t-1}) \to \widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^{2t}/\mathbb{R}\mathrm{P}^{2t-1}) \cong \widetilde{K}_{\mathbb{C}}(S^{2t}) \cong \mathbb{Z} \to 0$$

   forms a short exact sequence, with $\nu_2^{(t)}$ mapping to a generator of $\mathbb{Z}$. Such a short exact sequence always splits, so we get the first half of part 2 of the theorem:

   $$\widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{2t-1}) \cong \mathbb{Z} \oplus \widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{2t}).$$

All that remains is to prove that $\widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{2t}) \cong \mathbb{Z}_{2^f}$ and the two relations

$$\nu^2 = -2\nu \text{ and } \nu^{f+1} = 0,$$

(and the computation of the Adams operations). To do this, we first prove the relation $\nu^2 = -2\nu$. We know that $w_1 : \mathrm{Vect}^1_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n) \to H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2$ is an isomorphism.

It follows that $\gamma \otimes \gamma = \epsilon^1$ for all real line bundles γ on $\mathbb{R}\mathrm{P}^n$. In particular, $\xi^2 = \epsilon^1$ so $(\epsilon^1 + \lambda)^2 = \epsilon^1$. By expanding this,

$$\lambda^2 = -2\lambda. \tag{7.3}$$

Then $\nu^2 = c(\lambda^2) = -2c(\lambda) = -2\nu$, as required.

This relation resolves the question of extensions in the filtration of $\widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n)$ in point 1. above. As previously noted, in this case the generators of the quotients are $\nu, \nu^2, ..., \nu^f$. Then $\nu^2 = -2\nu$ forces the extension to be $\mathbb{Z}_{2^f}$. We illustrate this by the specific case $f = 2$: There are, naively, two possible extensions

1. $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by $(\nu, 0)$ and $(0, \nu^2)$, or
2. $\mathbb{Z}_4$.

The first possibility cannot satisfy the relation $\nu^2 = -2\nu$, so it must be the second extension. This argument extends to general f, since it shows that $\widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n)$ cannot be a product of groups.

Thus, $\widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}_{2^f}$. This forces the second relation in part 1.(a) of the theorem: $\nu^{f+1} = 0$. The general result $\widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{2t}) \cong \mathbb{Z}_{2^f}$ then follows by part 1.(b) of the theorem, which we proved above.

Finally, we compute the Adams operations $\Psi^k_{\mathbb{C}}$. We do the $m = 2t$ case first. From above, $\xi^2 = 1$ and so $c\xi^2 = 1$. As we know how Adams operations behave on line bundles,

$$\Psi^k_{\mathbb{C}}(c\xi) = \begin{cases} 1 & \text{if } k \text{ even,} \\ c\xi & \text{if } k \text{ odd.} \end{cases}$$

Since $\nu = c\xi - 1$, we have

$$\Psi^k_{\mathbb{C}}(\nu) = \begin{cases} 0 & \text{if } k \text{ even,} \\ \nu & \text{if } k \text{ odd,} \end{cases}$$

from which we obtain

$$\Psi^k_{\mathbb{C}}(\nu^s) = \begin{cases} 0 & \text{if } k \text{ even,} \\ \nu^s & \text{otherwise.} \end{cases}$$

The computation of $\Psi^k_{\mathbb{C}}(\nu_1^{(t+1)})$ then follows by naturality

$$q^*\Psi^k_{\mathbb{C}}(\nu_1^{(t+1)}) = \Psi^k_{\mathbb{C}}(\nu^{t+1}) = \begin{cases} \nu^{t+1} = q^*\nu_1^{(t+1)} & \text{if } k \text{ odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Since q^* is injective, the desired result follows.

Now we move onto the $m = 2t - 1$ case. From the structure $\widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{2t-1}) \cong \mathbb{Z} \oplus \widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{2t})$, we have

$$\Psi^k_{\mathbb{C}}(\nu_2^{(t)}) = a\nu_2^{(t)} + b\nu_1^{(t+1)},$$

for some $a \in \mathbb{Z}, b \in \mathbb{Z}_{2^f}$. Let $i : S^{2t} = \mathbb{R}\mathrm{P}^{2t}/\mathbb{R}\mathrm{P}^{2t-1} \to \mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{2t-1}$ be the injection. Then, remembering that $\nu_2^{(t)}$ doesn't depend on the index n,

$$a\nu_2^{(t)} = i^*\Psi_{\mathbb{C}}^k(\nu_2^{(t)}) = \Psi_{\mathbb{C}}^k(i^*\nu_2^{(t)}) = k^t i^*\nu_2^{(t)} = k^t\nu_2^{(t)}$$

where the second last equality comes from Corollary 1.48. Thus, $a = k^t$. To compute b, consider $q : \mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{2t-2} \to \mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{2t-1}$. By Lemma 7.8, $q^*\nu_2^{(t)} = \nu_1^{(t)}$. Also, $q^*\nu_1^{(t+1)} = (\nu_1^{(t)})^2 = -2\nu_1^{(t)}$, so

$$a\nu_1^{(t)} - 2b\nu_1^{(t)} = q^*\Psi_{\mathbb{C}}^k(\nu_2^{(t)}) = \Psi_{\mathbb{C}}^k(q^*\nu_2^{(t)}) = \Psi_{\mathbb{C}}^k(\nu_1^{(t)}),$$

in $\widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{2t-2}) \cong \mathbb{Z}_{2^{f+1}}$. If k is odd, we get

$$k^t\nu_1^{(t)} - 2b\nu_1^{(t)} = \Psi_{\mathbb{C}}^k(\nu_1^{(t)}) = \nu_1^{(t)},$$

and so $b = \frac{1}{2}(k^t - 1) \in \mathbb{Z}_{2^f}$. If k is even, we get $k^t\nu_1^{(t)} - 2b\nu_1^{(t)} = 0$ and so $b = \frac{1}{2}k^t \in \mathbb{Z}_{2^f}$. $\square$

7.3 The Real K-Theory of $\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m$

Define $\phi(n, m)$ to be the number of integers p such that $m < p \leq n$ and $p \equiv 0, 1, 2$, or $4 \mod 8$.

Theorem 7.10. *1. Suppose $m \not\equiv -1 \mod 4$. Then there is a group isomorphism $\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m) \cong \mathbb{Z}_{2^f}$ where $f = \phi(n, m)$.*

(a) The (non-unital) ring $\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n)$ is generated by λ with two relations, describing multiplication

$$\lambda^2 = -2\lambda \text{ and } \lambda^{f+1} = 0.$$

(As in the previous theorem, this implies $2^f\lambda = 0$.)

(b) For $m \neq 0$, the projection $q_1 : \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m$ induces an isomorphism from $\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$ onto the ideal of $\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n)$ generated by λ^{g+1}, where $g = \phi(m, 0)$.

2. Suppose $m \equiv -1 \mod 4$. Writing $m = 4t - 1$, we have

$$\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4t-1}) \cong \mathbb{Z} \oplus \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4t}),$$

where the second summand is embedded by the quotient $q_2 : \mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4t-1} \to \mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4t}$.

For clarity, we delay all proofs and present the results on the Adams operations. First we need to develop some notation for the generators. Let $m \not\equiv -1 \mod 4$ and $g = \phi(m, 0)$. Define $\lambda_1^{(g+1)}$ to be the element in $\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$ that is mapped to

$\lambda^{g+1} \in \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n)$ by q_1^*. Then the Adams operations on $\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$ are completely described by their value on $\lambda_1^{(g+1)}$.

Now let $m = 4t-1$ and define $g = \phi(4t, 0)$. Write $\lambda_2^{(g)}$ for a generator, which we define below Lemma 7.17 of the first summand in

$$\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4t-1}) \cong \mathbb{Z} \oplus \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4t}).$$

(Here $g = \phi(m, 0)$ as before.)

Theorem 7.11. *The Adams operations $\Psi_{\mathbb{R}}^k : \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m) \to \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$ for $m \not\equiv -1 \mod 4$ and $m \equiv -1 \mod 4$ are given by*

$$\Psi_{\mathbb{R}}^k \lambda_1^{(g+1)} = \begin{cases} \lambda_1^{(g+1)} & \text{if } k \text{ odd,} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\Psi_{\mathbb{R}}^k \lambda_2^{(g)} = k^{2t}\lambda_2^{(g)} + \begin{cases} \frac{1}{2}(k^{2t}-1)\lambda_1^{(g+1)} & \text{if } k \text{ odd,} \\ \frac{1}{2}k^{2t}\lambda_1^{(g+1)} & \text{otherwise,} \end{cases}$$

respectively, where $m = 4t-1$ in the second case.

To prove Theorem 7.10, we split it into a number of smaller, more manageable results. But first we need some preliminary lemmata. We begin by examining the relevant spectral sequence.

Lemma 7.12. *Let $(E_r^{*,*}, d_r)$ be the Atiyah Hirzebruch spectral sequence associated with $\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$. There are $\phi(n,m)$ non-zero groups along the main diagonal of the E_2 page. If $m \equiv -1 \mod 4$, then one of these groups is $\mathbb{Z}$ and all the others are $\mathbb{Z}_2$. Otherwise, they are all $\mathbb{Z}_2$.*

Unlike in Theorem 7.9, we will not show directly that these groups survive to the E_∞ page. Instead, we will leverage the results from Theorem 7.9, to get that they survive and that the extensions are trivial.

Proof. Recall real Bott periodicity from chapter 1:

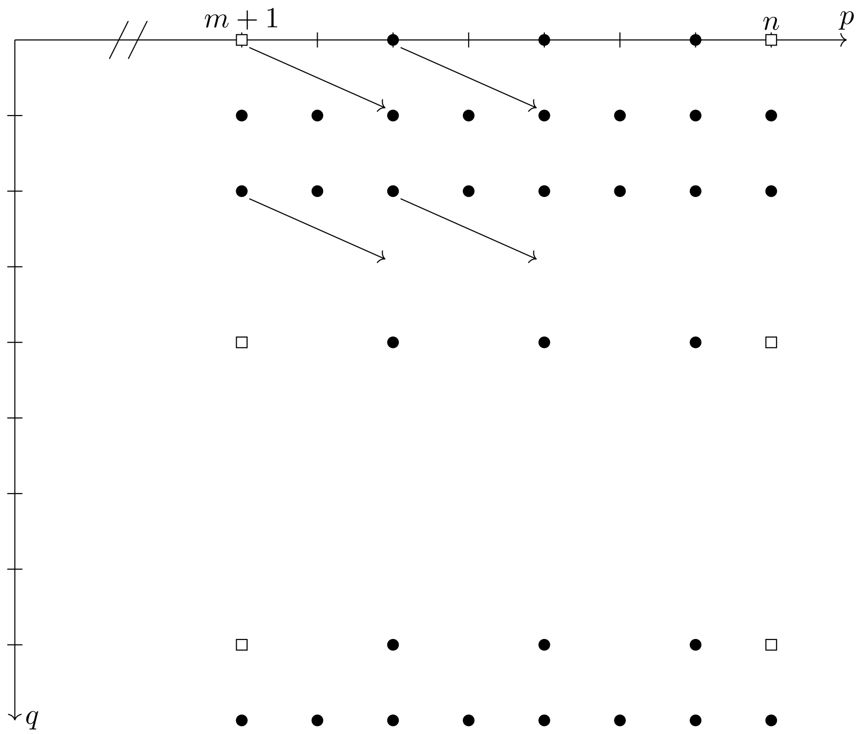
$$K_{\mathbb{R}}^q(*) \cong \begin{cases} \mathbb{Z}_2 & \text{if } q \equiv 6 \text{ or } 7 \mod 8, \\ \mathbb{Z} & \text{if } q \equiv 0 \text{ or } 4 \mod 8, \\ 0 & \text{otherwise.} \end{cases}$$

We compute the second page $E_2^{p,q} \cong \widetilde{H}^p\left(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m, K_{\mathbb{R}}^q(*)\right)$ using equations 7.1 and

7.2:

$q \equiv 7 \mod 8$	$\widetilde{H}^p\left(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m;\mathbb{Z}_2\right)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$q \equiv 6 \mod 8$	$\widetilde{H}^p\left(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m;\mathbb{Z}_2\right)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$q \equiv 4 \mod 8$	$\widetilde{H}^p\left(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m;\mathbb{Z}\right)$	$\mathbb{Z}$		$\mathbb{Z}_2$		$\mathbb{Z}_2$	$\mathbb{Z}$
$q \equiv 0 \mod 8$	$\widetilde{H}^p\left(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m;\mathbb{Z}\right)$	$\mathbb{Z}$		$\mathbb{Z}_2$		$\mathbb{Z}_2$	$\mathbb{Z}$
		even	odd	even	odd	even	odd
		$p = m+1$		$m+1 < p < n$		$p = n$	

As in the Theorem 7.9, we want to find the non-zero $E_\infty^{p,-p}$ terms and thus we are mostly concerned with the fourth quadrant. As an example, the fourth quadrant of the second page is drawn below, for n, m odd. The circles are $\mathbb{Z}_2$ and the squares are $\mathbb{Z}$.



By inspecting the E_2 page, we reach the following conclusions:

1. If $m+1 \not\equiv 0 \mod 4$, then the only non-zero terms along the main diagonal of E_2 are $\mathbb{Z}_2$. But $E_2^{p,-p}$ is non-zero if and only if $p \equiv 0, 1, 2$ or $4 \mod 8$ and $m+1 \leq p \leq n$. Therefore, there are $\phi(n,m)$ copies of $\mathbb{Z}_2$ on the main diagonal.

2. If $m+1 \equiv 0 \mod 4$ then $E_2^{m+1,-m-1} \cong \mathbb{Z}$. All the other non-zero terms along the main diagonal of E_2 are $\mathbb{Z}_2$. Moreover, as before, $E_2^{p,-p}$ is non-zero if and only if $p \equiv 0, 1, 2$ or $4 \mod 8$ and $m+1 \leq p \leq n$. Therefore, there are $\phi(n,m) - 1$ copies of $\mathbb{Z}_2$ and one copy of $\mathbb{Z}$ on the main diagonal. □

Lemma 7.13. *If* $n \equiv 0, 6$ *or* $7 \mod 8$ *then the complexification map*

$$c : \widetilde{K}_{\mathbb{R}}(\mathbb{R}P^n) \to \widetilde{K}_{\mathbb{C}}(\mathbb{R}P^n)$$

is an isomorphism.

Proof. We know that $\widetilde{K}_{\mathbb{C}}(\mathbb{R}P^n)$ is generated by ν by Theorem 7.9. By definition $\nu = c\lambda$, where $\lambda = \xi - \epsilon^1 \in \widetilde{K}_{\mathbb{R}}(\mathbb{R}P^n)$. Thus, $c : \widetilde{K}_{\mathbb{R}}(\mathbb{R}P^n) \to \widetilde{K}_{\mathbb{C}}(\mathbb{R}P^n)$ is always surjective.

By the above lemma, $\widetilde{K}_{\mathbb{R}}(\mathbb{R}P^n)$ has a filtration whose successive quotients are $\phi(n, 0)$ copies of $\mathbb{Z}_2$. It follows that number of elements in $\widetilde{K}_{\mathbb{R}}(\mathbb{R}P^n)$ is at most $2^{\phi(n,0)}$.

When $n = 8t + 6$, or $n = 8t + 7$, we have $\phi(n, 0) = 4t + 3$. So $\widetilde{K}_{\mathbb{R}}(\mathbb{R}P^n)$ contains at most 2^{4t+3} elements. But by Theorem 7.9, $\widetilde{K}_{\mathbb{C}}(\mathbb{R}P^n)$ contains exactly

$$2^{\lfloor \frac{1}{2}n \rfloor} = 2^{4t+3}$$

elements.

Since c is surjective, $\widetilde{K}_{\mathbb{R}}(\mathbb{R}P^n)$ must contain 2^{4t+3} elements and consequently, c is an isomorphism.

When $n = 8t$, a similar argument applies: $\phi(n, 0) = 4t$ and $\lfloor \frac{1}{2}n \rfloor = 4t$ so $\widetilde{K}_{\mathbb{R}}(\mathbb{R}P^n)$ and $\widetilde{K}_{\mathbb{C}}(\mathbb{R}P^n)$ have the same number of elements. □

Lemma 7.14. *Part 1.(a) of Theorem 7.10: There is a group isomorphism* $\widetilde{K}_{\mathbb{R}}(\mathbb{R}P^n) \cong \mathbb{Z}_{2^f}$ *where* $f = \phi(n, 0)$, *and as a ring* $\widetilde{K}_{\mathbb{R}}(\mathbb{R}P^n)$ *is generated by* λ *with two relations, describing multiplication*

$$\lambda^2 = -2\lambda \textit{ and } \lambda^{f+1} = 0.$$

Proof. Theorem 7.9 and the above lemma show that $\widetilde{K}_{\mathbb{R}}(\mathbb{R}P^n) \cong \mathbb{Z}_{2^f}$, for $n \equiv 0, 6$ or $7 \mod 8$. In this case, we also have that λ generates $\widetilde{K}_{\mathbb{R}}(\mathbb{R}P^n)$, since $c\lambda = \nu$ generates $\widetilde{K}_{\mathbb{C}}(\mathbb{R}P^n)$,

Now reconsider the spectral sequence $(E_r^{*,*}, d_r)$ for $\widetilde{K}_{\mathbb{R}}(\mathbb{R}P^n)$. Along the main diagonal of the E_2 page, there are $\phi(n, 0)$ copies of $\mathbb{Z}_2$. All other terms on the main diagonal are zero. If $n \equiv 0, 6$ or $7 \mod 8$, then all of the main diagonal terms on the E_2 page must survive to the E_∞ page. Otherwise, $\widetilde{K}_{\mathbb{R}}(\mathbb{R}P^n)$ could not be in bijection with $\widetilde{K}_{\mathbb{C}}(\mathbb{R}P^n)$.

We want to show that this is true for general n. Fix some n and choose $\hat{n} > n$ with $\hat{n} \equiv 0 \mod 8$. Let $(E_r^{*,*}, d_r)$ and $(\hat{E}_r^{*,*}, \hat{d}_r)$ be the spectral sequences associated with $\widetilde{K}_{\mathbb{R}}(\mathbb{R}P^n)$ and $\widetilde{K}_{\mathbb{R}}(\mathbb{R}P^{\hat{n}})$ respectively. Note that the E_2 page is (roughly) a truncated version of the $\hat{E}_2$ page. The inclusion map $\mathbb{R}P^n \to \mathbb{R}P^{\hat{n}}$ induces a map of spectral sequences $\hat{E}_r^{*,*} \to E_r^{*,*}$. It follows that if $\hat{d}_r$ is zero then the corresponding d_r is also zero. Therefore, all the main diagonal terms on the E_2 page survive to the E_∞ page. However, it is not immediately obvious that what the extensions of the filtration for $\widetilde{K}_{\mathbb{R}}(\mathbb{R}P^n)$ are.

Note that we already showed $\lambda^2 = -2\lambda$. See equation 7.3 in the proof of Theorem 7.9. Then by the same reasoning as in the proof of Theorem 7.9, the relation $\lambda^2 = -2\lambda$ resolves the question of extensions for general n. Therefore ,we have $\widetilde{K}_{\mathbb{R}}(\mathbb{R}P^n) \cong \mathbb{Z}_{2^f}$ generated by λ. The second relation $\lambda^{f+1} = 0$ follows from the first and the fact that$2^f\lambda = 0$. □

Lemma 7.15. *Part 1.(b) of Theorem 7.10: Suppose $m \not\equiv -1 \mod 4$ and $m \neq 0$. There is a group isomorphism $\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m) \cong \mathbb{Z}_{2^f}$ where $f = \phi(n,m)$. Moreover, the projection $q_1 : \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m$ induces an isomorphism from $\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$ onto the ideal of $\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n)$ generated by λ^{g+1}, where $g = \phi(m,0)$.*

Proof. Consider the exact sequence

$$\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m) \xrightarrow{q_1^*} \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n) \xrightarrow{i^*} \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^m).$$

We know that $\lambda \in \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n)$ is sent to $\lambda \in \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^m)$ by i^*. Therefore, i^* is surjective with kernel

$$\mathrm{Ker}\, i^* = \left\{0, 2^{\phi(m,0)}\lambda, 2 \times 2^{\phi(m,0)}\lambda, 3 \times 2^{\phi(m,0)}\lambda ..., (2^{\phi(n,0)-\phi(m,0)} - 1)2^{\phi(m,0)}\lambda\right\}.$$

Thus, $\mathrm{Ker}\, i^*$ has 2^f elements, where $f = \phi(n,0) - \phi(m,0) = \phi(n,m)$.

When $m \not\equiv -1 \mod 4$, there is a filtration on $\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$ such that the successive quotients are f copies of $\mathbb{Z}_2$. Thus, $\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$ has at most 2^f elements. By exactness, it follows that

1. $\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$ has exactly 2^f elements;
2. The map q_1^* is injective and therefore embeds $\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$ into $\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n)$ as the ideal $\mathrm{Ker}\, i^*$ generated by $2^{\phi(m,0)}\lambda = \pm\lambda^{\phi(m,0)+1}$. □

We have now proved part 1. of Theorem 7.10. We move onto part 2.

Lemma 7.16. *Part 2 of Theorem 7.10: Suppose $m \equiv -1 \mod 4$. Writing $m = 4t-1$, we have*

$$\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4t-1}) \cong \mathbb{Z} \oplus \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4t}),$$

where the second summand is embedded by the quotient $q_2 : \mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4t-1} \to \mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4t}$.

Proof. It suffices to show that

$$0 \to \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4t}) \xrightarrow{q_2^*} \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4t-1}) \xrightarrow{i_2^*} \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{4t}/\mathbb{R}\mathrm{P}^{4t-1}) \cong \widetilde{K}_{\mathbb{R}}(S^{4t}) \cong \mathbb{Z} \to 0$$

is a split short exact sequence, where i_2^* is the obvious inclusion map. Such a short exact sequence always splits, so we need only show that q_2^* is injective and i^* is surjective.

In Lemma 7.15, we showed that the map

$$\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4t}) \xrightarrow{q_1^*} \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n)$$

is injective. Since q_2^* factors through q_1^*, it follows that q_2^* is also injective.

We will now show that i_2^* is surjective. Consider the commutative diagram, with the bottom row and the middle and right columns exact and every map above is induced by an inclusion or a projection, except δ which is the connecting homomorphism in the relevant long exact sequence.

$$\begin{tikzcd}
0 \cong \widetilde{K}_{\mathbb{R}}(S^{4t-1}) = \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{4t-1}/\mathbb{R}\mathrm{P}^{4t-2}) \arrow[r, equal] & \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{4t-1}/\mathbb{R}\mathrm{P}^{4t-2}) \\
\mathbb{Z} \cong \widetilde{K}_{\mathbb{R}}(S^{4t}) = \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{4t}/\mathbb{R}\mathrm{P}^{4t-2}) \arrow[u] & \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{n}/\mathbb{R}\mathrm{P}^{4t-2}) \arrow[l, "i_1^*"'] \arrow[u] \\
\widetilde{K}^1_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{n}/\mathbb{R}\mathrm{P}^{4t}) \xleftarrow{\ \delta\ } \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{4t}/\mathbb{R}\mathrm{P}^{4t-1}) \arrow[u, "j_1^*"] & \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{n}/\mathbb{R}\mathrm{P}^{4t-1}). \arrow[l, "i_2^*"'] \arrow[u, "j_2^*"']
\end{tikzcd}$$

We immediately see that j_1^* and j_2^* are surjective, since $\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{4t-1}/\mathbb{R}\mathrm{P}^{4t-2}) \cong 0$. By Lemma 7.15, $\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{4t}/\mathbb{R}\mathrm{P}^{4t-2}) \cong \mathbb{Z}_2$ and i_1^* is a surjection.

Thus $j_1^* i_2^* = i_1^* j_2^*$ is a surjection. But j_1^* is a surjection from

$$\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{4t}/\mathbb{R}\mathrm{P}^{4t-1}) = \widetilde{K}_{\mathbb{R}}(S^{4t}) \cong \mathbb{Z}$$

to $\mathbb{Z}_2$, which means j_1^* is the map $z \mapsto z \mod 2$. It follows that some odd number is in the image of i_2^*. Let a be the smallest positive odd number in the image of i_2^*. Then $a \in \operatorname{Ker}\delta$ and so $\mathbb{Z}_a$ embeds in $\operatorname{Im}\delta$. This implies $\delta(1)$ has odd order.

Recall the spectral sequence for $\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^4 t)$. We know $\widetilde{K}^1_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4t})$ is built from the groups $E_\infty^{p,1-p}$, which are in turn built out of the groups $E_2^{p,1-p}$. But all $E_2^{p,1-p}$ are either zero, $\mathbb{Z}_2$ or $\mathbb{Z}$. This means that there are no elements of odd order in $E_\infty^{p,1-p}$, except zero, and therefore, no elements of odd order in $\widetilde{K}^1_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4t})$, except zero. Thus, $\delta(1) = 0$, which implies $\operatorname{Im} i_2^* = \operatorname{Ker}\delta = \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{4t}/\mathbb{R}\mathrm{P}^{4t-1})$. So i_2^* is surjective, as required. $\square$

We have now proven Theorem 7.10. Before proving Theorem 7.11, we need to construct the generator $\lambda_2^{(g)}$. To do this, we state the following result whose proof requires material beyond the scope of this thesis.

Lemma 7.17 [1, Lemma 7.7]. *If $n \equiv 0, 6$ or $7 \mod 8$ then the complexification map*

$$c : \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/RP^{4t-1}) \to \widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4t-1})$$

is an isomorphism for t even and a monomorphism for t odd.

In the case $n \equiv 0, 6$ or $7 \mod 8$, we define $\lambda_2^{(g)} \in \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4t-1})$ as follows. If $t = 2u$ even, then $g = \phi(4t, 0) = 4u$ and we can define $\lambda_2^{(g)}$ to be the unique element that is mapped to $\nu_2^{(g)}$ by c. If $t = 2u + 1$ odd, then $g = 4u + 3$ and we define $\lambda_2^{(g)} = -r\nu_2^{(4u+2)}$, where r is the 'realification' map. With some work (c.f. proof of lemma 7.7 of [1]), one obtains $i_2^*\lambda_2^{(g)} \in \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{4t}/\mathbb{R}\mathrm{P}^{4t-1})$. So we indeed have that $\lambda_2^{(g)}$ is a generator of $\mathbb{Z}$ in

$$\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4t-1}) \cong \mathbb{Z} \oplus \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4t}).$$

If $n \not\equiv 0, 6$ or $7 \mod 8$, we take some $\hat{n} > n$ with $\hat{n} \equiv 0 \mod 8$. Define $\lambda_2^{(g)} \in \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4t-1})$ as the image of $\lambda_2^{(g)} \in \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{\hat{n}}/\mathbb{R}\mathrm{P}^{4t-1})$ under the map induced by the inclusion. This definition doesn't depend on the choice of $\hat{n}$ since, as we decrease $\hat{n}$, each $\lambda_2^{(g)}$ map into each other.

Note that the image of $\lambda_2^{(g)}$ under the map

$$\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4t-1}) \xrightarrow{c} \widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4t-1}) \xrightarrow{q^*} \widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n)$$

is ν^g. Since c commutes with q^*, the generator $\lambda_2^{(g)}$ maps to $\lambda^g \in \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n)$ under $q^* : \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4t-1}) \to \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n)$. This explains the choice of notation.

We can now compute the Adams operations:

Proof of Theorem 7.11. To start with, assume $m \not\equiv -1 \mod 4$. The method in the proof of Theorem 7.9 is also applicable here: we know $\xi^2 = 1$ and we know how Adams operations behave on line bundles, so

$$\Psi_{\mathbb{R}}^k(\xi) = \begin{cases} 1 & \text{if } k \text{ even,} \\ \xi & \text{if } k \text{ odd.} \end{cases}$$

Then we can compute the Adams operations for $\lambda = \xi - 1$:

$$\Psi_{\mathbb{R}}^k(\lambda) = \begin{cases} 0 & \text{if } k \text{ even,} \\ \lambda & \text{if } k \text{ odd,} \end{cases}$$

from which we obtain

$$\Psi_{\mathbb{R}}^k(\lambda^s) = \begin{cases} 0 & \text{if } k \text{ even,} \\ \lambda^s & \text{otherwise.} \end{cases}$$

The computation of $\Psi_{\mathbb{R}}^k(\lambda_1^{(t+1)})$ then follows by naturality

$$q_1^*\Psi_{\mathbb{R}}^k(\lambda_1^{(t+1)}) = \Psi_{\mathbb{R}}^k(\lambda^{t+1}) = \begin{cases} \lambda^{t+1} = q_1^*\lambda_1^{(t+1)} & \text{if } k \text{ odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Since q_1^* is injective, the desired result follows.

Now assume $m = 4t-1$. We will again leverage the known results for $\widetilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$ by passing through the complexification map c. Specifically, if $n \equiv 0, 6$ or $7 \mod 8$, then we know by Lemmata 7.13 and 7.17 that c is injective. Moreover, the complexification map commutes with the Adams operations. Thus, in the case $t = 2u$ even, we have $g = 2t$ and

$$\begin{aligned} c\Psi_{\mathbb{R}}^k(\lambda_2^{(g)}) = \Psi_{\mathbb{C}}^k(\nu_2^{(g)}) &= k^g\nu_2^{(g)} + \begin{cases} \frac{1}{2}k^g\nu_1^{(g+1)} & \text{if } k \text{ even,} \\ \frac{1}{2}(k^g - 1)\nu_1^{(g+1)} & \text{otherwise.} \end{cases} \\ &= k^{2t}c(\lambda_2^{(g)}) + \begin{cases} \frac{1}{2}k^{2t}c(\lambda_1^{(g+1)}) & \text{if } k \text{ even,} \\ \frac{1}{2}(k^{2t} - 1)c(\lambda_1^{(g+1)}) & \text{otherwise.} \end{cases} \end{aligned}$$

as required. The case when t is odd follows similarly.

Finally, if $n \not\equiv 0, 6$ or $7 \mod 8$, then we can embed $\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$ into $\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{\hat{n}}/\mathbb{R}\mathrm{P}^m)$, for some $\hat{n} > n$ with $\hat{n}$ divisible by 8. The Adams operations on $\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$ then follow by naturality. $\square$

Chapter 8

Proof of the Main Theorems

In this chapter, we prove the two main theorems (1.5 and 1.3), thereby resolving the vector fields problem.

8.1 Proof of Theorem 1.5

Recall that we defined $n = (2a+1)2^b, b = c + 4d$ and $\rho(n) = 2^c + 4d$. In this section we prove Theorem 1.5, which states that $\mathbb{R}\mathrm{P}_n^{n+\rho(n)}$ is not coreducible, for $d \neq 0$. Since we have proved the main result (theorem 1.3) using Steenrod squares when $d = 0$, we are not concerned with the coreducibility of $\mathbb{R}\mathrm{P}_n^{n+\rho(n)}$ when $d = 0$.

The following is one attempt at proving Theorem 1.5: If $S^n \xrightarrow{i} \mathbb{R}\mathrm{P}_n^{n+k} \xrightarrow{f} S^n$ is degree 1, then the induced map on ordinary cohomology

$$\mathbb{Z} \cong H^n(S^n;\mathbb{Z}) \leftarrow H^n(\mathbb{R}\mathrm{P}_n^{n+k};\mathbb{Z}) \leftarrow H^n(S^n;\mathbb{Z}) \cong \mathbb{Z}$$

must be the identity. If $n > 0$ is odd and $k \geq 1$, then $H^n(\mathbb{R}\mathrm{P}_n^{n+k}) = 0$. But the identity map $\mathbb{Z} \to \mathbb{Z}$ does not factor through 0. Thus, $\mathbb{R}\mathrm{P}_n^{n+k}$ cannot possibly be coreducible.

For n not odd, we need a stronger test to prove that $\mathbb{R}\mathrm{P}_n^{n+k}$ is not coreducible. It turns out that K-theory provides this stronger test. That is, we will show that

$$\widetilde{K}_{\mathbb{R}}(S^n) \xleftarrow{i^*} \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}_n^{n+\rho(n)}) \xleftarrow{f^*} \widetilde{K}_{\mathbb{R}}(S^n)$$

cannot be equality, by using the results on stunted projective spaces from the chapter 7. Specifically, we will use the fact that $\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}_n^{n+\rho(n)})$ splits as a direct sum, with one summand generating $\widetilde{K}_{\mathbb{R}}(S^n)$. Then we will use Adams operations to derive a contradiction.

To prove this theorem, we will need the following easy number-theoretic result, whose proof is delayed until the end of this section.

Lemma 8.1. *If $m = (2a'+1)2^{b'}$ with $b' \geq 1$, then $3^m - 1 \equiv 2^{b'+2} \mod 2^{b'+3}$.*

Proof of Theorem 1.5. Since $d \neq 0$, we know $n \equiv 0 \mod 8$. Recall the function $\phi(p,q)$ whose value is the number of integers s such that $q < s \leq p$ and $s \equiv 0, 1, 2$ or $4 \mod 8$.

We compute

$$\phi\left(n+\rho(n),n\right) = \phi\left(\rho(n),0\right) = \phi(8d,0)+\phi(2^c,0) = 4d+c+1 = b+1.$$

By Theorem 7.10,

$$\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{n+\rho(n)}/\mathbb{R}\mathrm{P}^{n-1}) = \mathbb{Z}\oplus\mathbb{Z}_{2^{b+1}}$$

where the first summand is generated by $\lambda_2^{(g+1)}$ and the second by $\lambda_1^{(g+1)}$, with $g=\frac{1}{2}n$. Recall that the second summand is an embedding $j^*:\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{n+\rho(n)}/\mathbb{R}\mathrm{P}^{n}) \to \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{n+\rho(n)}/\mathbb{R}\mathrm{P}^{n-1})$ induced by the inclusion. From the split exact sequence in the proof of Lemma 7.16,

$$\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{n+\rho(n)}/\mathbb{R}\mathrm{P}^{n}) \xrightarrow{j^*} \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{n+\rho(n)}/\mathbb{R}\mathrm{P}^{n-1}) \xrightarrow{i^*} \widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{n}/\mathbb{R}\mathrm{P}^{n-1}) \cong \mathbb{Z},$$

we can conclude that $i^*\lambda_1^{(g+1)}=0$ and $i^*\lambda_2^{(g+1)}$ is a generator γ of $\widetilde{K}_{\mathbb{R}}(S^n)\cong\mathbb{Z}$.

Now suppose, for contradiction, that we had a map $f:\mathbb{R}\mathrm{P}^{n+\rho(n)}/\mathbb{R}\mathrm{P}^{n-1}\to S^n$ such that the composition with i has degree 1. Then f^*i^* is multiplication by one and so $f^*i^*\gamma=\gamma$. Therefore,

$$f^*\gamma = N\lambda_1^{(g+1)}+\lambda_2^{(g+1)}$$

for some integer N.

By naturality of the Adams operations,

$$f^*\Psi_{\mathbb{R}}^k\gamma = \Psi_{\mathbb{R}}^k f^*\gamma = \Psi_{\mathbb{R}}^k\left(N\lambda_1^{(g+1)}+\lambda_2^{(g+1)}\right),$$

for all integers k.

We know how $\Psi_{\mathbb{R}}^k$ acts on S^n by Corollary 1.48 and on $\mathbb{R}\mathrm{P}^{n+\rho(n)}/\mathbb{R}\mathrm{P}^{n-1}$ by Theorem 7.11, so we get

$$f^*(k^{n/2}\gamma) = k^{m/2}\lambda_2^{(g+1)}+\frac{1}{2}\left(k^{m/2}-\delta\right)\lambda_1^{(g+1)}+\delta N\lambda^{(g+1)},$$

where $\delta=0$ if k even and 1 otherwise. Then

$$k^{m/2}N\lambda_1^{(g+1)}+k^{m/2}\lambda_2^{(g+1)} = k^{m/2}\lambda_2^{(g+1)}+\frac{1}{2}\left(k^{m/2}-\delta\right)\lambda_1^{(g+1)}+\delta N\lambda^{(g+1)}$$

which factors to

$$\left(N-\frac{1}{2}\right)\left(k^{m/2}-\delta\right)\lambda_1^{(g+1)}=0.$$

Since $\lambda_1^{(g+1)}$ is a generator of $\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{n+\rho(n)}/\mathbb{R}\mathrm{P}^{n})\cong\mathbb{Z}_{2^{b+1}}$, it follows

$$\left(N-\frac{1}{2}\right)\left(k^{m/2}-\delta\right)\equiv 0 \mod 2^{b+1},$$

or

$$(2N-1)(k^{m/2}-\delta)\equiv 0 \mod 2^{b+2},$$

for all k. Now Lemma 8.1 gives us a contradiction in the case $k = 3$: taking $m = \frac{n}{2}$ in the lemma, we have $b' = b - 1$ and so

$$3^{n/2} - 1 \equiv 2^{b+1} \mod 2^{b+2}$$

which implies

$$M(3^{n/2} - \delta) \equiv 2^{b+1} \not\equiv 0 \mod 2^{b+2}$$

for any odd M. $\square$

Proof of Lemma 8.1. Firstly, we prove by induction on b' that

$$3^{2^{b'}} - 1 \equiv 2^{b'+2} \mod 2^{b'+3},$$

for $b' \geq 1$. In the case $b' = 1$, the result is true since $3^{2^1} - 1 = 8$. In the step case, we have

$$3^{2^{b'+1}} - 1 = \left(3^{2^{b'}} - 1\right)\left(3^{2^{b'}} + 1\right).$$

By the induction hypothesis, $3^{2^{b'}} - 1 = 2^{b'+2} + q2^{b'+3}$ for some integer q. Also, since $3^2 \equiv 1 \mod 8$, we have $3^{2^{b'}} \equiv 1 \mod 8$ and consequently $3^{2^{b'}} + 1 = 2 + r2^3$ for some integer r. Thus,

$$3^{2^{b'+1}} - 1 = \left(2^{b'+2} + q2^{b'+3}\right)\left(2 + r2^3\right) \equiv 2^{b'+3} \mod 2^{b'+4},$$

which proves the induction step.

Now, we have $\left(3^{2^{b'}} - 1\right)^2 \equiv 0 \mod 2^{b'+3}$ by this induction result and consequently

$$3^{2 \cdot 2^{b'}} \equiv 2(3^{2^{b'}} - 1) + 1 \equiv 2^{b'+3} + 1 \equiv 1 \mod 2^{b'+3}.$$

Then $3^{(2a')2^{b'}} \equiv 1 \mod 2^{b'+3}$ so

$$3^m - 1 \equiv 3^{2^{b'}} \cdot 3^{(2a')2^{b'}} - 1 \equiv 3^{2^{b'}} - 1 \equiv 2^{b'+2} \mod 2^{b'+3},$$

where the last equivalence follows by using the induction result. $\square$

8.2 Proof of Theorem 1.3

We need one final result before we can prove Theorem 1.3.

Theorem 8.2. *In the stable homotopy category, for $n \geq 0$,*

$$\Sigma^{-k}\mathbb{R}\mathrm{P}_k^{n+k}$$

only depends on k modulo $2^{\phi(n,0)}$, where $\phi(n, m)$ is the number of integers p such that $m < p \leq n$ and $p \equiv 0, 1, 2,$ or $4 \mod 8$.

This theorem is called James periodicity, after I. M. James.

An immediate corollary, is that

$$\Sigma^{-k}\mathbb{R}\mathrm{P}_k^{n+k} \cong \Sigma^{-k-qf}\mathbb{R}\mathrm{P}_{k+qf}^{n+k+qf},$$

where $f = 2^{\phi(n,0)}$, and consequently

$$\mathbb{R}\mathrm{P}_k^{n+k} \cong \Sigma^{-qf}\mathbb{R}\mathrm{P}_{k+qf}^{n+k+qf},$$

for all integers q.

Proof. We may suppose that $n > 0$, since for $n = 0$, the theorem is a tautology. We know that $\mathbb{R}\mathrm{P}_k^{n+k} = \mathrm{Th}(\mathbb{R}\mathrm{P}^n, k\xi)$ only depends on the class of $k\xi$ in $K_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n)$ by Proposition 4.34. Equivalently, $\Sigma^{-k}\mathrm{Th}(\mathbb{R}\mathrm{P}^n, k\xi)$ only depends on $k\xi - \epsilon^k$ in $\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n)$.

Recall from Theorem 7.10 that $\lambda = \xi - \epsilon^1$ is the generator of $\widetilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n)$ and has order $2^{\phi(n,0)}$. Thus, $k\xi - \epsilon^k = k\lambda$ depends only on k modulo $2^{\phi(n,0)}$. □

Theorem 1.3. *There does not exists $\rho(n)$ vector fields on S^{n-1}.*

We will prove this by contradiction, using Theorem 1.5 proved above.

Proof. Since we have already proved this result when n is not a multiple of 16 in chapter 3, we may assume that $d \neq 0$, where $n = (2a+1)2^b$ with $b = c + 4d$.

Suppose there did exist $\rho(n)$ vector fields on S^{n-1}. Then there exists $\rho(n)$ vector fields on S^{pn-1} by Lemma 2.12, for all p. By Lemma 1.40, there exists a section $V_{pn,\rho(n)+1} \to S^{pn-1}$. We chose p to be odd and large enough such that $pn - 1 \leq 2(pn - \rho(n) - 1)$. (The fact we need p odd will only become apparent at the end of the proof.) Then we can apply Corollary 1.42. We obtain a map $S^{pn-1} \to \mathbb{R}\mathrm{P}_{pn-\rho(n)-1}^{pn-1}$ giving a homotopy equivalence

$$S^{pn-1} \vee \mathbb{R}\mathrm{P}_{pn-\rho(n)-1}^{pn-2} \simeq \mathbb{R}\mathrm{P}_{pn-\rho(n)-1}^{pn-1}.$$

It follows that there is a weak equivalence between the suspension spectra of the above spaces. By taking Spanier-Whitehead duals, which preserves wedge sums, we have

$$S^{-pn+1} \vee \Sigma\mathbb{R}\mathrm{P}_{-pn+1}^{-pn+\rho(n)} \cong \Sigma\mathbb{R}\mathrm{P}_{-pn}^{-pn+\rho(n)},$$

where we have compute $D\mathbb{R}\mathrm{P}_{pn-\rho(n)-1}^{pn-2}$ using Theorem 5.17.

Since Σ is an equivalence in spectra, we can take its inverse to get a weak equivalence

$$S^{-pn} \vee \mathbb{R}\mathrm{P}_{-pn+1}^{-pn+\rho(n)} \cong \mathbb{R}\mathrm{P}_{-pn}^{-pn+\rho(n)}.$$

We now want to apply James periodicity so that we can move into the stable range. We know that $\mathbb{R}\mathrm{P}_{-pn}^{-pn+\rho(n)}$ only depends on $-pn$ modulo $f = 2^{\phi(\rho(n),0)}$ and $\mathbb{R}\mathrm{P}_{-pn+1}^{-pn+\rho(n)}$ only depends on $-pn+1$ modulo $f' = 2^{\phi(\rho(n)-1,0)}$. Since either $f = f'$ or $f = 2f'$, we have that $\mathbb{R}\mathrm{P}_{-pn+1}^{-pn+\rho(n)}$ only depends on $-pn+1$ modulo f. Thus,

$$S^{-pn} \vee \Sigma^{-qf}\mathbb{R}\mathrm{P}_{qf-pn+1}^{qf-pn+\rho(n)} \cong \Sigma^{-qf}\mathbb{R}\mathrm{P}_{qf-pn}^{qf-pn+\rho(n)},$$

or

$$S^{qf-pn} \vee \mathbb{R}\mathrm{P}^{qf-pn+\rho(n)}_{qf-pn+1} \cong \mathbb{R}\mathrm{P}^{qf-pn+\rho(n)}_{qf-pn},$$

for all q.

We choose q so that:

1. We have $qf - pn > 0$. This ensures that $\mathbb{R}\mathrm{P}^{qf-pn+\rho(n)}_{qf-pn} \cong \mathbb{R}\mathrm{P}^{qf-pn+\rho(n)}/\mathbb{R}\mathrm{P}^{qf-pn-1}$ is actually a stunted projective space and S^{qf-pn} a sphere.

2. We have $\rho(n) + 1 < qf - pn$. By Corollary 4.44, this ensures that we are in the stable range. Then the weak equivalence

$$S^{qf-pn} \vee \mathbb{R}\mathrm{P}^{qf-pn+\rho(n)}_{qf-pn+1} \cong \mathbb{R}\mathrm{P}^{qf-pn+\rho(n)}_{qf-pn},$$

corresponds to a homotopy equivalence

$$S^{qf-pn} \vee \mathbb{R}\mathrm{P}^{qf-pn+\rho(n)}_{qf-pn+1} \simeq \mathbb{R}\mathrm{P}^{qf-pn+\rho(n)}_{qf-pn},$$

of spaces.

3. We have qf is a multiple of $2n$.

Therefore, the space $\mathbb{R}\mathrm{P}^{qf-pn+\rho(n)}/\mathbb{R}\mathrm{P}^{qf-pn-1}$ is coreducible. Since p is odd and qf is a multiple of $2n$, it follows that $m = qf - pn$ is an odd multiple of n. So $\rho(m) = \rho(n)$. Moreover, m is divisible by 16, since n is. Thus,

$$\mathbb{R}\mathrm{P}^{qf-pn+\rho(n)}/\mathbb{R}\mathrm{P}^{qf-pn-1} = \mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}$$

is not coreducible. But this contradicts Theorem 1.5. $\square$

Appendix A

Some Category Theory

A.1 Limits

Definition A.1. Let I be a (small) category. A *diagram of shape I* is a functor $D : I \to \mathscr{C}$.

A *cone* of a diagram D is an object $A \in \mathscr{C}$ with a collection of maps $(A \xrightarrow{f_i} D(i))_{i \in I}$.

A *limit* of a diagram D is a cone $(L, \{f_i\}_{i\in I})$ such that, for every cone $(A, \{f_i'\}_{i\in I})$, there exists a unique map $\overline{f} : A \to L$ with

$$\begin{tikzcd} & A \arrow[dl, "f_i'"'] \arrow[d, dashed, "\overline{f}"] \\ D(i) & L \arrow[l, "f_i"] \end{tikzcd}$$

commuting, for all $i \in I$.

Definition A.2. Let I be a (small) category. A *cocone* of a diagram $D : I \to \mathscr{C}$ is an object $A \in \mathscr{C}$ with a collection of maps $(D(i) \xrightarrow{f_i} A)_{i\in I}$.

A *colimit* of a diagram D is a cocone $(L, \{f_i\}_{i\in I})$ such that, for every cocone $(A, \{f_i'\}_{i\in I})$, there exists a unique map $\overline{f} : L \to A$ with

$$\begin{tikzcd} & A \\ D(i) \arrow[ur, "f_i'"] \arrow[r, "f_i"'] & L \arrow[u, dashed, "\overline{f}"'] \end{tikzcd}$$

commuting, for all $i \in I$.

Direct Limits

Definition A.3. Let I be the category

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

A *directed system* is a diagram of shape $I^\dagger$. A *direct limit* (or *inductive limit*) is the colimit of a directed system. If

$$A_1 \to A_2 \to A_3 \to \dots$$

is a directed system, define $\operatorname{colim}_{n\to\infty} A_n$ to be the associated direct limit.

Proposition A.4. *Modules (and therefore abelian groups) always have direct limit.*

We omit the proof of this proposition.

Proposition A.5. *Suppose we have a sequence of sets with maps*

$$S_1 \to S_2 \to S_3 \to \dots$$

that are injective. Then the direct limit of this sequence exists.

Proof idea. Define an relation on $\bigcup_i S_i$ by the following $x \sim y$ if $x \in S_i$ and $y \in S_j$ with $i < j$ and x gets mapped to y by $S_i \to S_{i+1} \to \dots \to S_j$. Take the reflexive symmetric closure of $\sim$. Then $\sim$ is an equivalence relation on $\bigcup_i S_i$. The set of equivalence classes satisfies the universal property. □

This proves that the direct limit

$$C = \operatorname*{colim}_{n\to\infty} C_n$$

in Definition 4.35 of a stable cell always exists.

A.2 Monoidal Categories

Definition A.6. A *monoidal category* is a tuple $(\mathscr{C}, \otimes, a, S, \iota)$ where

1. $\mathscr{C}$ is a category,
2. $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ is a bifunctor, callend the *tensor product*,
3. $a : (- \otimes -) \otimes - \xrightarrow{\cong} - \otimes (- \otimes -)$ is a natural isomorphism, called the *associator*,
4. $S \in \mathscr{C}$, called the *unit object*.

We require a monoidal category to satisfy two axioms:

1. the pentagon axiom (c.f. [8, §2.1]), and
2. the unit axiom: there are natural isomorphisms $\lambda_X : S \otimes X \cong X$ and $\rho_X : X \otimes S \cong X$, called the *left* and *right unitors* respectively.

A monoidal category is *associative* if its associator a is the identity. A monoidal category is *strict* if its associator and its left and right unitors are the identity.

A monoidal category is *symmetric* if there exists a natural isomorphism $X \otimes Y \xrightarrow{\cong} Y \otimes X$.

†Actually, a directed system is more general than this: let I be a small category with a preorder (reflexive and transitive relation) on objects such that every pair has an upper bound. Further, the morphisms of I consist of $f_{ij} : i \to j$ for all $i \le j$ such that $f_{ii} = \mathsf{id}_i$ and $f_{jk} \circ f_{ij} = f_{ik}$ for all $i \le j \le k$. A directed system is a diagram of shape I.

A.3 Duals

Definition A.7. Suppose $\mathscr{C}$ is a monoidal category with unit object I. A dual of an object $X \in \mathscr{C}$ is an object X^* with maps

$$\epsilon : X \otimes X^* \to S \text{ and } \eta : S \to X \otimes X^*,$$

such that the two pentagon diagrams commute (c.f. [8, §2.10]).

If $\mathscr{C}$ is an associative, symmetric monoidal category, then the pentagon axioms can be restated as requiring

$$X \otimes S \xrightarrow{1\otimes\eta} X \otimes X^* \otimes X \xrightarrow{\epsilon\otimes 1} S \otimes X \text{ and } X^* \otimes S \xrightarrow{1\otimes\eta} X^* \otimes X \otimes X^* \xrightarrow{\epsilon\otimes 1} S \otimes X^*$$

to be the canonical isomorphisms.

Theorem A.8. *1. If X^* is dual to X, then for all Y and W, the map*

$$\mathrm{Hom}(Y \otimes X^*, W) \xrightarrow{-\otimes X} \mathrm{Hom}(Y \otimes X^* \otimes X, W \otimes X) \xrightarrow{-\circ\eta\circ\rho_Y^{-1}} \mathrm{Hom}(Y, W \otimes X)$$

is an isomorphism, where ρ_Y is the natural isomorphism $Y \otimes S \cong Y$.

2. If X^* is a dual of X, then X is a dual of X^*.

A.4 Internal Hom

Definition A.9. Let $\mathscr{C}$ be a monoidal category. An *internal hom* in $\mathscr{C}$ is a functor

$$[-,-] : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$$

such that for every object $X \in \mathscr{C}$, the functors $- \otimes X$ and $[X, -]$ are adjoint:

$$- \otimes X \dashv [X, -].$$

If an *internal hom* exists, then $\mathscr{C}$ is called *closed*.

In a closed category, there is a isomorphism

$$\mathrm{Hom}(X, [Y, Z]) \xrightarrow{\cong} \mathrm{Hom}(X \otimes Y, Z),$$

which is natural in all three variables.

The terminology 'closed' refers to the fact that forming hom-sets does not lead 'out of the category'. The stable homotopy category is closed symmetric monoidal category.

Bibliography

[1] J. F. Adams. Vector fields on spheres. *Annals of Mathematics*, 75:603–632, 1962.

[2] J. F. Adams. *Stable Homotopy and Generalised Homology.* Chicago Lectures in Mathematical Series, The University of Chicago Press, Chicago, 1974.

[3] M. Atiyah. Thom complexes. *Proceedings of the London Mathematical Society (3)*, 11:291–310, 1961.

[4] M. F. Atiyah and F. Hirzebruch. Vector bundles and homogeneous spaces. *Differential Geometry*, 3:7–38, 1961.

[5] L. E. J. Brouwer. Über abbildung von mannigfaltigkeiten. *Mathematische Annalen*, 71:97–115, 1912.

[6] B. Eckmann. Gruppentheoretischer beweis des satzes von Hurwitz-Radon über die komposition quadratischer formen. *Commentarii Mathematici Helvetici*, 15:358–366, 1942.

[7] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, Modules, and Algebras in Stable Homotopy Theory*, volume 47 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.

[8] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik. *Tensor Categories.* American Mathematical Society, Providence, 2015.

[9] A Hatcher. *Algebraic Topology.* Cambridge University Press, Cambridge, 2002.

[10] A. Hatcher. *Vector Bundles and K Theory.* (unpublished), version 2.1 (2009) edition, 2003. retrieved from `https://www.math.cornell.edu/~hatcher/VBKT/VBpage.html` on 7/02/17.

[11] A Hatcher. *Spectral Sequences in Algebraic Topology.* (unpublished), 2004. retrieved from `https://www.math.cornell.edu/~hatcher/SSAT/SSATpage.html` on 25/01/17.

[12] M. Hovey, B. Shipley, and J. Smith. Symmetric spectra. *Journal of the American Mathematical Society*, 13:149–208, 2000.

[13] I. M. James. Spaces associated with Stiefel manifolds. *Proceedings of the London Mathematical Society (3)*, 9:115–140, 1959.

[14] I. M. James. *The Topology of Stiefel Manifolds.* Cambridge University Press, Cambridge, 1977.

[15] R. C. Kirby and L. C. Siebenmann. On the triangulation of manifolds and the hauptvermutung. *Bulletin of the American Mathematical Society*, 75(742-749), 1969.

[16] A. A. Kosinski. *Differential Manifolds.* Academic Press, 1993.

[17] E. L. Lima. *Duality and Postnikov Invariants.* PhD thesis, University of Chicago, 1958.

[18] J. McCleary. *A User's Guide to Spectral Sequences.* Cambridge University Press, 2nd edition, 2001.

[19] A. R. Rajwade. *Squares.* London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1993.

[20] E. H. Spanier. *Algebraic Topology.* McGraw-Hill, 1966.

[21] N. E. Steenrod and J. H. C. Whitehead. Vector fields on n-spheres. *Proceedings of National Academy Science, USA*, pages 58–63, 1951.

[22] H. Toda. Vector fields on spheres. *Bulletin of American Mathematical Society*, 67:408–412, 1961.

[23] Rainer Vogt. *Boardman's Stable Homotopy Category.* Number 21 in Lecture note series. Matematisk Institut, Aarhus Universitet, 1970.

[24] J. H. C. Whitehead. On the groups $\pi_r(v_{n,m})$ and sphere-bundles. *Proceedings of the London Mathematical Society (2)*, 49:479–481, 1947.

[25] K. Wickelgren. *Stable Homotopy Theory Lecture Notes.* Georgia Institute of Technology, 2015. retrieved from `http://people.math.gatech.edu/~kwickelgren3/Teachingwebpage.html` on 23/06/17.

[26] Wikimedia. *Hairy_ball.png.* 15/03/2007. retrieved from `https://upload.wikimedia.org/wikipedia/commons/e/ec/Hairy_ball.png` on 30/09/17.