A Künneth Formula for Complex K Theory

James Bailie

June 6, 2017

1 Introduction

A Künneth formula relates the (co)homology of two objects to the (co)homology of their product. For example, there is a split short exact sequence

$$0 \to \bigoplus_i \left(H_i(X;R) \otimes_R H_{n-i}(Y;R)\right) \to H_n(X \times Y;R) \to \bigoplus_i \mathrm{Tor}_R\left(H_i(X;R), H_{n-i-1}(Y;R)\right) \to 0$$

for CW complexes X and Y and prinicipal ideal domains R [AT, theorem 3B.6]. A similar result for cohomology (with added assumptions) can be found in [Sp, theorem 5.11].

In this report we will prove a Künneth formula for complex K theory:

Theorem 1. *There is a natural exact sequence*

$$0 \to \bigoplus_{i+j=k} K^i(X) \otimes K^j(Y) \xrightarrow{\mu} K^k(X \times Y) \xrightarrow{\beta} \bigoplus_{i+j=k+1} \mathrm{Tor}\left(K^i(X), K^j(Y)\right) \to 0$$

for finite CW complexes X and Y, where all indices are in $\mathbb{Z}_2$ and μ is the external product.

This theorem was first proven by Atiyah in 1962 [VBKF].

Sections 2 to 4 provide some necessary background to the proof of theorem 1. Section 5 contains the proof. There is a brief discussion on the impossibility of a Künneth formula for real K theory in seciton 6. In section 7 we provide a stronger Künneth formula, given by Atiya in [KT]. Finially we will briefly discuss Kunneth formulae for other general cohomology theories in section 8.

2 Review of the Tor Functor

Recall the definition of Tor given by Hatcher in [AT, section 3.A]: from an abelian group H and a free resolution F:

$$... \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

we can form a chain complex by removing H and tensoring with a fixed abelian group G:

$$... \to F_2 \otimes G \xrightarrow{f_2 \otimes \mathsf{id}} F_1 \otimes G \xrightarrow{f_1 \otimes \mathsf{id}} F_0 \otimes G \to 0.$$

The homology groups $H_n(F \otimes G)$ of this chain complex do not depend on the free resolution F [AT, lemma 3.A2]. We denote these groups by $\mathrm{Tor}_n(H, G)$.

We can generalise this to the case when G and H are R-modules. This generalisation is necessary to understand the Künneth formula for ordinary homology given above. However, it is inessential for our purposes since in the Künneth formula for K theory, Tor is applied to abelian groups.

Note that for any abelian group H, a free resolution of the form

$$0 \to F_1 \to F_0 \to H \to 0$$

always exists. We can construct such a resolution as follows. Let $\alpha_1, ..., \alpha_n$ (n possibly infinite) be generators for H and define F_0 to be the free abelian group over $\alpha_1, ..., \alpha_n$ (considered as formal symbols). There is a surjective homomorphism $f_0 : F_0 \to H$ defined by

$$f_0(\alpha_i) = \alpha_i$$

and extended linearly. Now the kernel of f_0 is free, since it is a subgroup of a free abelian group. Therefore, we obtain a free resolution

$$0 \to \mathrm{Ker}\, f_0 \overset{i}{\hookrightarrow} F_0 \xrightarrow{f_0} H \to 0,$$

with the following associated chain complex

$$0 \to \mathrm{Ker}\, f_0 \otimes G \xrightarrow{i \otimes \mathsf{id}} F_0 \otimes G \to 0.$$

The homology groups $H_n(F \otimes G)$ of this chain complex are obviously 0 for $n > 1$. Since these homology groups do not depend on the choice of free resolution, it must be the case that $H_n(F \otimes G) = 0$ for all free resolutions F of H and all $n > 1$. In other words, $\mathrm{Tor}_n(H, G) = 0$ for $n > 1$. Moreover, since the tensor product is right exact, $\mathrm{Tor}_0(H, G) \cong H \otimes G$. Therefore, when H is an abelian group, there is only one Tor group of interest, $\mathrm{Tor}_1(H, G)$. This is what we mean when we write $\mathrm{Tor}(H, G)$ without any subscript. In the more general case when G and H are R-modules, the higher Tor groups do not vanish, and hence must be included in the Künneth formula, as we have seen for homology with coefficients in a principal idea domain.

Finally, it is worth noting that $\mathrm{Tor}(H, G)$ is a covariant functor of both G and H. Given a homomorphism $\alpha : H \to H'$ between abelian groups, there is an induced homomorphism $\alpha_* : \mathrm{Tor}(H, G) \to \mathrm{Tor}(H', G)$ on Tor groups, defined in the following way. Firstly, note that given free resolutions F and F' of H and H' respectively, we can extend α to a chain map:

$$\begin{array}{ccccccccc}
... & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & & \\
... & \longrightarrow & F_2' & \xrightarrow{f_2'} & F_1' & \xrightarrow{f_1'} & F_0' & \xrightarrow{f_0'} & H' & \longrightarrow & 0
\end{array}$$

This is lemma 3.1(a) in [AT]. Then tensoring resolutions with H, we get a chain map $\alpha_n \otimes \mathsf{id}$. This chain map induces a homomorphism $\alpha_* : H_n(F \otimes G) \to H_n(F' \otimes G)$ on homology. Note that this map satisfies the required functorial properties: $(\alpha\alpha')_* = \alpha_*\alpha'_*$ and $\mathsf{id}_* = \mathsf{id}$. The construction of the induced map $\beta_* : \mathrm{Tor}(H, G) \to \mathrm{Tor}(H, G')$ from a homomorphism $\beta : G \to G'$ is obvious.

3 Classifying Spaces

To understand Atiyah's proof of theorem 1, we need some knowledge classifying spaces. This section assumes some comprehension of homotopy theory - chapter 4 of [AT] is sufficient.

Definition 2 (informal)**.** Let G be a topological group. A *principal G bundle* is a fibre bundle $p : E \to B$, where E is a G-space and each fibre is homeomorphic to G.

Recall that all n-dimensional vector bundles are pullbacks of the universal bundle $E_n(\mathbb{C}^\infty) \to G_n(\mathbb{C}^\infty)$. There is an analogous result for principal G bundles [Mitchell, theorem 7.4 and theorem 7.6]: We have a prinicipal G-bundle

$$p : EG \to BG$$

called the universal bundle with the property that a principal G bundle over any G-space X is a pullback of p by some map $f : X \to BG$.

Definition 3. We call BG the *classifying space* of G.

It turns out that there is an obvious fibration:

$$G \hookrightarrow EG \to BG.$$

Then using the following proposition and the fact that EG is contractible, we get that there is a weak homotopy equivalence $G \to \Omega BG$, where ΩBG is the loopspace of BG. (This tells us that the classifying space BG is unique up to (at least) weak homotopy equivalence.)

Proposition 4 [AT, proposition 4.66]**.** *If $F \to E \to B$ is a fibration with E contractible, then there is a weak homotopy equivalence $F \to \Omega B$.*

This obviously strengthens to homotopy equivalence, when considering CW complexes. The idea behind the proof is to construct maps between fibrations

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow{=} \\ \Omega B & \longrightarrow & PB & \longrightarrow & B \end{array}$$

where PB is the path space of B. Then consider the long exact sequences of these two fibrations, with maps between them induced by the diagram above. Since E and PB are contractible, their homotopy groups vanish and the five lemma implies that the map $F \to \Omega B$ is a weak homotopy equivalence.

We are now in a position to give some important results that will be used in the proof of the main theorem. Let $U(n)$ be the unitary group of degree n and U be the infinite unitary group. Denote the Grassmannian of n-dimensional subspaces of V by $G_n(V)$. We have that the classifying space $BU(n) \cong G_n(\mathbb{C}^\infty)$ [Mitchell, section 7]. Thus, there are obvious inclusions $BU(n) \hookrightarrow BU(n+1) \hookrightarrow BU$.

Proposition 5. *Let $[A, B]$ be the homotopy classes of maps $A \to B$. Then $\widetilde{K}^0(X) \cong [X, BU]$.*

Proof. A class $\alpha \in \widetilde{K}(X)$ is represented by $[E] - [\epsilon^n]$ where $\dim E = n$ and is classified by a map $f : X \to G_n(\mathbb{C}^\infty)$ [VBKT, theorem 1.16]. But α is also represented by $[E \oplus \epsilon^1] - [\epsilon^{n+1}]$ and therefore classified by a map $f' : X \to G_{n+1}(\mathbb{C}^\infty)$. Now, we claim that two vector bundles classified by $g : X \to G_m(\mathbb{C}^\infty)$ and $g' : X \to G_{m'}(\mathbb{C}^\infty)$ are stably equivalent (i.e. they represent the same class in $\widetilde{K}(X)$) if and only if the maps

$$X \xrightarrow{g} G_m(\mathbb{C}^\infty) \hookrightarrow G_k(\mathbb{C}^\infty) \text{ and } X \xrightarrow{g'} G_{m'}(\mathbb{C}^\infty) \hookrightarrow G_k(\mathbb{C}^\infty)$$

are homotopic for some large k. This proves the claim. ∎

Corollary 6. *If X is CW complex, then $K^1(X) \cong [X, U]$.*

Proof. By definition
$$K^1(X) = \widetilde{K}^0(\Sigma X) \cong [\Sigma X, BU],$$
where ΣX is the reduced suspension of X. Using the adjoint relation between Σ and the loopspace Ω (c.f. [AT, section 4.3]), we have $[\Sigma X, BU] \cong [X, \Omega BU]$. Since X is a CW complex, we have $\Omega BU \simeq U$ by proposition 4, which proves the claim. □

See also [KT, lemma 2.4.6] for a proof of corollary 6 with a different flavour.

4 Atiyah-Hirzebruch Spectral Sequence

In this section, we provide an informal description of a spectral sequence and state some results necessary for proving the Künneth formula for complex K theory.

Spectral sequences are useful tools for calculating (co)homology and homotopy groups. A spectral sequence is a three dimensional array $E^r_{p,q}$ of abelian groups with $p, q, r \in \mathbb{Z}, r > 0$. For fixed r, the groups $E^r_{p,q}$ are called the r-th page. (Often the r-th page is denoted simply by E^r.) There are morphisms $d_r : E^r_{p,q} \to E_{p-r,q-r+1}$ which form chain complexes. That is, we have that $d_r^2 = 0$ and consequently we call d_r a differential. Finally, the $(r+1)$-th page is the homology of the chain complexes in the r-th page:

$$E^{r+1}_{p,q} = \frac{\mathrm{Ker}\,(d_r : E^r_{p,q} \to E^r_{p-r,q+r-1})}{\mathrm{Im}(d_r : E^r_{p+r,q-r+1} \to E^r_{p,q})}.$$

Often, we can assume $E^r_{p,q} = 0$ for $p, q < 0$. Then, for fixed p, q and large enough r, the differentials going in and out of $E^r_{p,q}$ will be zero. At this point, passing to the next page will not change the groups: $E^{r+1}_{p,q} = E^r_{p,q}$. So the groups stabilise and it follows that there is a well defined limit page $E^\infty_{p,q}$. In general, we say that a spectral sequence converges if for every p, q the differential d_r vanishes on $E^r_{p,q}$ and $E_{p+r,q-r+1}$, for large enough r. In cases where $E^r_{p,q} \neq 0$ for $p, q < 0$, convergence can be guaranteed by other means.

We will now give a short example computation using spectral sequences. The Serre spectral sequence is defined for fibrations $F \to X \to B$ and relates the homology of F, X and B. The second page is given by $E^2_{p,q} = H_p(B; H_q(F; G))$ where G is a given coefficient group. If G is a field then $H_n(X; G) \cong \oplus_p E^\infty_{p,n-p}$. If $G = \mathbb{Z}$ and $H_i(F; \mathbb{Z})$ and $H_i(B; \mathbb{Z})$ are zero for odd i and free abelian for even i, then it follows that $E^2_{p,q}$ is zero unless p and q are even. But the differentials on this page go up one row, so they all must be zero. So the E^3 page must be the same as E^2. The differentials on the E^3 page go across three columns, hence they must all be zero. Similar reasoning applies for the subsequent pages, so $E^2 = E^3 = E^r = E^\infty$. Thus, $E^\infty_{p,q} = H_p(B; H_q(F; G))$. Since these groups are free abelian, it is possible to deduce that $H_n(X; \mathbb{Z}) \cong \oplus_p E^\infty_{p,n-p} = \oplus_p H_p(B; H_{n-p}(F; G))$ as in the case when G was a field. By the universal coefficient theorem, this is isomorphic to $\oplus_p H_p(B; H_{n-p}(F; G))$. Note that if X was the product $F \times B$, the Künneth formula also tells us that $H_n(X; \mathbb{Z}) \cong \oplus_p H_p(B; H_{n-p}(F; G))$, since $\mathrm{Tor}(G, H)$ vanishes if G or H is free. Hopefully, this particularly nice example demonstrates the power of spectral sequences.

Spectral sequences are also useful for computing K theory. In particular, we have the following theorem, due to Atiyah and Hirzebruch. Suppose X is a finite CW complex and let X^p be the

p-skeleton of X with inclusion $X^p \overset{i}{\hookrightarrow} X$. Let N be the dimension of X. Define a (descending) filtration

$$K^n(X) = K_0^n(X) \supset K_1^n(X) \supset ... \supset K_{N+1}^n(X) = 0$$

of $K^n(X)$ by $K_p^n(X) = \text{Ker}\left(K^n(X) \xrightarrow{i^*} K^n(X^{p-1})\right)$ for $p > 0$.

Theorem 7 [VBHS, theorem 2.1]**.** *Let X be a finite CW complex, as above, with basepoint x_0. There exists a spectral sequence $E_{p,q}^r$ with*

$$\begin{aligned} E_{p,q}^1 &\cong C^p\left(X, K^q(x_0)\right), \\ E_{p,q}^2 &\cong H^p\left(X, K^q(x_0)\right), \\ E_{p,q}^\infty &\cong K_p^{p+q}(X)/K_{p+1}^{p+q}(X). \end{aligned}$$

Note $K^q(x_0) \cong \mathbb{Z}$ for even q and zero otherwise. This theorem generalises from K theory to any general cohomology theory and $E_{p,q}^r$ is called the Atiyah-Hirzebruch spectral sequence.

We can also generalise to fiber bundles $F \to E \xrightarrow{p} X$, under certain conditions [VBHS, remark 2.2]. In particular, if we have the product bundle $X \times F$, then we have a spectral sequence $E_{p,q}^r$ with

$$\begin{aligned} E_{p,q}^1 &\cong C^p\left(X, K^q(F)\right)^\dagger, \\ E_{p,q}^2 &\cong H^p\left(X, K^q(F)\right), \\ E_{p,q}^\infty &\cong K_p^{p+q}(X \times F)/K_{p+1}^{p+q}(X \times F), \end{aligned}$$

where $K_p^n(X \times F) = \text{Ker}\left(K^n(X \times F) \xrightarrow{i^*} K^n(X^{p-1} \times F)\right)$ is a filtration of $K^n(X)$.

The proofs of these statements is beyond this report. The interested reader is refered to [VBHS] for the (historical) proof and [SSAT] for a modern, accessible introduction to spectral sequences.

5 Proof of the Main Theorem

Define $K^*(X) = K^0(X) \oplus K^1(X)$ and similarly define $\widetilde{K}^*(X)$. Throughout this section, let X and Y be finite CW complexes.

Lemma 8. *Suppose $K^*(Y)$ is free. Then*

$$K^*(X) \otimes K^*(Y) \xrightarrow{\mu} K^*(X \times Y)$$

is an isomorphism.

Proof. Use the spectral sequence $E_{p,q}^r$ in theorem 7 and the corresponding spectral sequence $F_{p,q}^r$ for the product $X \times Y$. By definition of local coefficients, we have an isomorphism

$$\mu_1 : E_1 \otimes K^*(Y) = C^*(X, \mathbb{Z}) \otimes K^*(Y) \longrightarrow C^*\left(X; K^*(Y)\right) = F_1.$$

Since the tensor product is natural with respect to the differentials [CM, lemma 3.1], this extends to maps

$$\mu_r : E_r \otimes K^*(Y) \to F_r.$$

†Here, we are using local coefficients, c.f. [AT, section 3.H].

Furthermore, since $K^*(Y)$ is free, the operation $\otimes K^*(Y)$ commutes with the homology functor. Therefore, μ_r is an isomorphism for all r.

In particular, μ_r is an isomorphism on the stable pages E^∞ and F^∞. Therefore, we have isomorphisms between the filtrations of $K^*(X)\otimes K^*(Y)$ and $K^*(X\times Y)$. Descending induction on these filtrations gives the required result. □

Lemma 9. *We can embed X as a subcomplex of a finite CW complex A so that*

1. *$K^*(A)$ is free and*
2. *$K^*(A)\to K^*(X)$ is surjective.*

Proof. It suffices to prove this lemma for connected X. In this case we have

$$K^*(X)\cong \mathbb{Z}\oplus \widetilde{K}^0(X)\oplus K^1(X).$$

Since X is a finite CW complex, $K^*(X)$ is a finitely-generated group. By reasoning in section 3, we have

$$\widetilde{K}^0(X)\cong [X,BU] \text{ and } K^1(X)\cong [X,U].$$

Choose homogeneous generators $\{x_i\}$ for $K^*(X)$. For each i choose finite approximations to BU or U by defining

$$A_i=\begin{cases}\frac{U(2n)}{U(n)\times U(n)} & \text{if } |x_i|=0,\\ U(n) & \text{if } |x_i|=1,\end{cases}$$

where $n>\dim X$ is some large integer. Define $A=\Pi A_i$.

By universality of $BU(n)$, for each x_i with $|x_i|=0$, we can choose an element $a_i\in\widetilde{K}^0(A_i)$ and a map $f_i:X\to A_i$ such that $x_i=f_i^*(a_i)$. Similarly, for x_i with $|x_i|=1$, there is an element $a_i\in K^1(A_i)$ and a map $f_i:X\to A_i$ such that $x_i=f_i^*(a_i)$.

Combine these maps to get $f:X\to A$. By construction, every generator of $K^*(X)$ is a pullback of some element in $K^*(A)$ by f. Thus, the enduced map f^* is surjective.

The cohomology $H^*(A_i;\mathbb{Z})$ is free. Using a result in [VBHS, section 2], it follows that $K^*(A)$ is free. Finally, we still require f to be an embedding of X into A. While this is not true, we can replace f by a homotopically equivalent inclusion and properties 1. and 2. will be preserved. □

Proof of theorem 1. Construct A as in lemma 9 and define $B=A/X$. Let b_0 be the canonical basepoint of B. By proposition 2.9 in [VBKT], we have a short exact sequence

$$0\to\widetilde{K}^*(B)\to K^*(A)\to K^*(X)\to 0, \tag{1}$$

where exactness at $K^*(X)$ follows from property 2. of lemma 9. Moreover, $K^*(A)$ is free, so $\widetilde{K}^*(B)$ is free and hence $K^*(B)\cong\widetilde{K}^*B)\oplus\mathbb{Z}$.

We can apply lemma 8 to find isomorphisms

$$K^*(A)\otimes K^*(Y)\cong K^*(A\times Y) \tag{2}$$

$$K^*(B)\otimes K^*(Y)\cong K^*(B\times Y). \tag{3}$$

We know $K^*(B\times Y,b_0\times Y)=\text{Ker }(K^*(B\times Y)\to K^*(b_0\times Y))$ by [VBHS, section 1.5]. Combining these last two results gives an isomorphism

$$\widetilde{K}^*(B)\otimes K^*(Y)\cong K^*(B\times Y,b_0\times Y). \tag{4}$$

Consider the exact triangle of the pair $(A \times Y, X \times Y)$:

$$\begin{array}{ccc} K^*(A \times Y, X \times Y) & \xrightarrow{\phi} & K^*(A \times Y) \\ & {}_{\beta}\nwarrow \quad \swarrow_{\theta} & \\ & K^*(X \times Y), & \end{array}$$

where ϕ and θ have degree 0 and β degree 1. Since $(A \times Y)/(X \times Y) = (B \times Y)/(b_0 \times Y)$, we can use equations 2 and 4 to rewrite this triangle as

$$\begin{array}{ccc} \widetilde{K}^*(B) \otimes K^*(Y) & \xrightarrow{\phi} & K^*(A) \otimes K^*(Y) \\ & {}_{\beta}\nwarrow \quad \swarrow_{\theta} & \\ & K^*(X \times Y). & \end{array}$$

This exact triangle gives us a short exact sequence

$$0 \to \operatorname{Coker} \phi \xrightarrow{\mu} K^*(X \times Y) \xrightarrow{\beta} \operatorname{Ker} \phi \to 0.$$

Notice that equation 1 is actually a free resolution of $K^*(X)$. Thus,

$$0 \to \widetilde{K}^*(B) \otimes K^*(Y) \xrightarrow{\phi} K^*(A) \otimes K^*(Y) \to 0$$

is a chain complex associated to $\operatorname{Tor}(K^*(X), K^*(Y))$. Therefore, $\operatorname{Ker} \phi \cong \operatorname{Tor}(K^*(X), K^*(Y))$.

Again using equation 1 and right-exactness of the tensor product, we obtain an exact sequence

$$K^*(B) \otimes K^*(Y) \xrightarrow{\phi} K^*(A) \otimes K^*(Y) \to K^*(X) \otimes K^*(Y) \to 0.$$

It follows $\operatorname{Coker} \phi \cong K^*(X) \otimes K^*(Y)$. Therefore, we have the required short exact sequence:

$$0 \to K^*(X) \otimes K^*(Y) \xrightarrow{\mu} K^*(X \times Y) \xrightarrow{\beta} \operatorname{Tor}(K^*(X), K^*(Y)) \to 0.$$

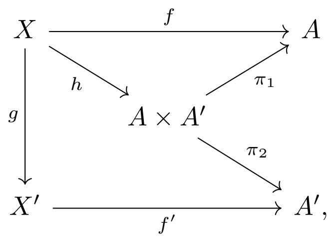
The $\mathbb{Z}_2$ grading is clear: by construction μ is degree 0 and β degree 1.

Finally, we prove that this exact sequence is natural. Naturality of μ is immediate from the construction of μ (see [VBKT, p.41]). Naturality of β is proven as follows: It is a category theory fact that we can verify naturality in X and Y by checking naturality in each of X and Y separately. We claim naturality in Y is obvious. For naturality in X, let $g : X \to X'$ be a given map. We need to show that the square

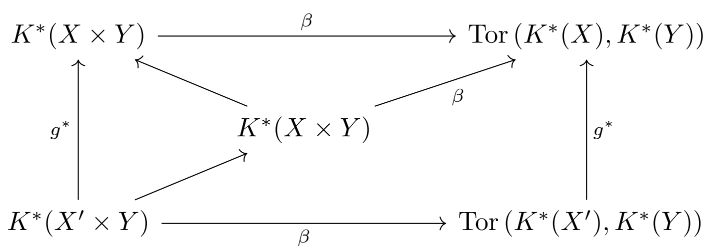
$$\begin{array}{ccc} K^*(X \times Y) & \xrightarrow{\beta} & \operatorname{Tor}(K^*(X), K^*(Y)) \\ \uparrow{\scriptstyle g^*} & & \uparrow{\scriptstyle g^*} \\ K^*(X' \times Y) & \xrightarrow{\beta} & \operatorname{Tor}(K^*(X'), K^*(Y)) \end{array}$$

commutes, where g^* are the obvious induced maps.

Construct spaces A and A' with maps $f : X \to A$ and $f' : X' \to A'$ as in lemma 9. Define the map $h = (f, f' \circ g) : X \to A \times A'$. Notice that h also satisfies properties 1. and 2. of lemma 9. We have the following commutative diagram:



where π_1, π_2 are projection maps. By applying the relevant functors, this diagram gives rise to the commutative diagram:



as required. (Moreover, it is worth noting that taking $X' = X$ shows that our short exact sequence is independent of the choice of f.) $\square$

6 A Künneth Formula for Real K Theory?

It is natural to ask whether there is an analogous Künneth formula for real K theory. It turns out that this can't happen, since in KO-theory the external product μ is not always injective. This implies there is no short exact sequence starting with

$$0 \to KO^*(X) \otimes KO^*(Y) \xrightarrow{\mu} KO^*(X \times Y)$$

and so the Künneth formula does not hold in the real case.

The counterexample given by Atiyah in [VBKF] is $X = Y = \mathbb{C}\mathrm{P}^2$. We will provide some justification why μ is not injective in the case $X = Y = S^2$. Take the non-trivial class $\alpha \in KO(S^2) \cong \mathbb{Z}_2$. It has order 2, so the tensor product $\alpha \otimes \alpha$ has order 2. Hence if μ is injective, $\mu(\alpha \otimes \alpha)$ has order 2 and should lie in the 4-cell of $S^2 \times S^2$. But there is no element of order two in $KO(S^4) \cong \mathbb{Z}$. Thus, μ cannot be injective.

7 A Stronger Künneth Formula

In [KT], Atiyah presents a stronger Künneth formula for complex K theory:

Theorem 10 ([KT, corollary 2.7.15])**.** *Let X be a space such that $K^*(X)$ is finitely generated and let Y be a finite CW complex. Then there is a natural short exact sequence*

$$0 \to \bigoplus_{i+j=k} K^i(X) \otimes K^j(Y) \to K^k(X \times Y) \to \bigoplus_{i+j=k+1} \operatorname{Tor}\left(K^i(X), K^j(Y)\right) \to 0,$$

where indices are in $\mathbb{Z}_2$.

This is a corollary of the following theorem, the proof of which is beyond the scope of this report.

Theorem 11 ([KT, theorem 2.7.15])**.** *Let X be a space such that $K^*(X)$ is torsion free, and let Y be a finite CW complex, with $Y_0 \subset Y$ a subcomplex. Then the external product*

$$K^*(X) \otimes K^*(Y, Y') \to K^*(X \times Y, X \times Y_0)$$

is an isomorphism.

Note that theorem 11 is a generalisation of lemma 8.

Proof (sketch) of theorem 10. The idea is to find a space A and a map $f : X \to A$ such that $K^*(A)$ is torsion free and $f^* : K^*(A) \to K^*(X)$ is surjective.

Suppose we have such a space A and a map f. Then by considering the exact sequence

$$... \to \widetilde{K}(SA) \to \widetilde{K}(SX) \to \widetilde{K}(A/X) \to \widetilde{K}(A) \to \widetilde{K}(X) \to 0,$$

it follows that $K^*(A/X)$ is torsion free. Applying theorem 11, we get that

$$K^*(A \times Y) \cong K^*(A) \otimes K^*(Y) \text{ and } K^*\left((A/X) \times Y\right) \cong K^*(A/X) \otimes K^*(Y).$$

Finally, use the exact sequence for the pair $(A \times Y, X \times Y)$

$$... \to \widetilde{K}\left(S(A \times Y)\right) \to \widetilde{K}\left(S(X \times Y)\right) \to \widetilde{K}\left((A \times Y)/(X \times Y)\right) \to \widetilde{K}(A \times Y) \to \widetilde{K}(X \times Y) \to 0,$$

to get the result, in the same way as in the proof of the theorem 1.

The construction of A and f is a generalisation of lemma 9 and is detailed in [KT, p.118-119]. □

8 Extending to a General Theorem for any Cohomology Theory

In this section, we will briefly discuss whether the Künneth formula holds for other general cohomology theories. To even formulate the Künneth formula, we need to have a meaningful notion of products in the cohomology theory. In this case, all cohomology groups would be modules over the graded ring $h^*(\mathrm{pt})$ (the cohomology of a point) and $\otimes, \operatorname{Tor}$ would be applied to $h^*(\mathrm{pt})$-modules.

We saw that the Künneth formula does not hold for all general cohomology theories in section 6. In the original paper [VBKF] proving the main theorem, Atiyah wrote:

> Thus for a Künneth formula to hold one has to assume some additional special property of the [cohomology] theory. The proof which we shall give for K^* is quite different from the proof for H^*. We shall use the fact that $H^*(BU, \mathbb{Z})$ is free and of course the analogue in ordinary cohomology is false, since the Eilenberg-MacLane spaces have torsion†.

†Eilenberg-MacLane spaces $K(G, n)$ are the analogue to the classifying space BU since $K(X) \cong [X, \mathbb{Z} \times BU]$ and $H^n(X, G) \cong [X, K(G, n)]$.

However, since this paper was published in 1962, a general proof method for Künneth formulae has been developed using spectral sequences. See sections 4 and 5 of [SHT].

Finally, we note that the Künneth formula relates the cohomologies of Z, X and Y when Z is the product of X and Y. The space Z can also be constructed from X and Y in other ways and spectral sequences can again be used to related the (co)homologies of X, Y and Z. For ordinary cohomology, the important result is:

Theorem 12 [SSAT, theorem 5E.2]**.** *Let k be a field. Suppose there are maps $X \to B$ and $Y \to B$ where the latter is a fibration. Let Z be the pullback*

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow \\ X & \longrightarrow & B. \end{array}$$

Then there is a spectral sequence with second page

$$E^2_{p,q} = \mathrm{Tor}^{H^*(B;k)}_{p,q}(H^*(X;k), H^*(Y;k))^\dagger$$

converging to $H^*(Z;k)$ if B is simply-connected and the cohomology groups of X, Y and B are finitely generated over k in each dimension.

This spectral sequence is called the Eilenberg-Moore spectral sequence.

References

[KT] M. F. Atiyah (notes by D. W. Anderson), *K-Theory, Lecture Notes*, W. A. Benjamin, New York, 1967.

[VBKF] M. F. Atiyah, *Vector Bundles and the Künneth Formula*, Topology **1** (1962), 245-248.

[CM] M. F. Atiyah and F. Hirzebruch, *Analytic Cycles on Complex Manifolds*, Topology **1** (1961), 22-45.

[VBHS] M. F. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, Differential Geometry **3** (1961), 7-38.

[SHT] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J.P. May, *Rings, Modules and Algebras in Stable Homotopy Theory*, American Mathematical Society, Providence, 1997.

[AT] A. Hatcher, *Algebraic Topology*, free online edition, Cambridge University Press, Cambridge, 2002 (updated March 2017) (available online at `https://www.math.cornell.edu/~hatcher/AT/ATpage.html`).

[SSAT] A. Hatcher, *Spectral Sequences*, 2004 (available online at `https://www.math.cornell.edu/~hatcher/#SSAT`).

[VBKT] A. Hatcher, *Vector Bundles and K Theory*, 2003 (updated May 2009) (available online at `https://www.math.cornell.edu/~hatcher/VBKT/VBpage.html`).

† If R is a graded ring and A and B are graded modules over R, then there is an induced grading on $\mathrm{Tor}^R_n(A,B)$. The mth graded subgroup of $\mathrm{Tor}^R_n(A,B)$ is then denoted $\mathrm{Tor}^R_{n,m}(A,B)$. See [SSAT, p.622].

[HW] P. J. Hilton and S. Wylie, *Homology Theory: An Introduction to Algebraic Topology*, Cambridge University Press, Cambridge, 1967.

[Hutchings] M. Hutchings, *Introduction to spectral sequences*, 2011 (available online at `https://math.berkeley.edu/~hutching/teach/215b-2011/ss.pdf`).

[Milnor] J. Milnor, *Construction of universal bundles, II*, Annals of Mathematics **63** (1956), 430-436.

[Mitchell] S. A. Mitchell, *Notes on principal bundles and classifying spaces*, 2001 (available online at `https://www3.nd.edu/~mbehren1/18.906/prin.pdf`).

[Sp] E.H. Spanier, *Algebraic Topology*, Springer, New York, 1966.