Stable Homotopy Theory and Category of Spectra

James Bailie
Supervised by Dr. Vigleik Angeltveit
Australian National University

Australian Government
Department of Education and Training

1 Introduction

Algebraic topology is the study of geometric shapes, called spaces, using tools from abstract algebra. The ultimate goal of topology is to categorise spaces in the following way(Gowers, 2008, pp. 383): Topologists say that two spaces are the same, or *homeomorphic*, when one space can be continuously deformed into the other. (The technical definition of a *homeomorphism* is a continuous function with a continuous inverse.) Categorising spaces in this way has turned out to be a very hard problem. Algebraic topology's contribution to this problem is to provide a number of algebraic invariants that homeomorphic spaces share. Examples of such invariants are homotopy groups, homology and cohomology.

In this report, we develop a special type of homotopy groups, called stable homotopy groups. The ground work of stable homotopy theory was laid in (Freudenthal, 1937) and further developed by Adams (see Adams (1974)). We then develop a number of types of spectra and discuss this topic in a categorical setting. Much of this work was also done by Adams, but has been improved upon by others in the last few decades. Finally, we conclude by presenting Brown's representability theorem, a surprising result connecting comohology with homotopy.

Knowledge of undergraduate level point-set topology and group theory is assumed. In addition, section 6 requires some basic category theory and section 7 requires knowledge of general cohomology theories. Chapter 0 of (Hatcher, 2002) would help with ease of comprehension but is not strictly necessary. Unless otherwise explicitly stated, all maps are continuous. X, Y denote topological spaces. We write (X, x_0) and (Y, y_0) for space with base points x_0 and y_0 respectively. $f : (X, x_0) \to (Y, y_0)$ is a base point preserving map. Sometimes we leave the base points implicit, when it is obvious by the context.

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2 Homotopy

An important concept in the process of building algebraic tools from topological spaces is the idea of homotopy.

Definition 2.1. A *homotopy* between basepoint preserving maps $f, g : (X, x_0) \to (Y, y_0)$ is a family of maps $F_t : (X, x_0) \to (Y, y_0)$ for $t \in [0, 1]$ such that

- $F_t(x)$ is continuous in both t and x,
- $F_0 = f$, and
- $F_1 = g$.

f and g are said to be *homotopic* if there is a homotopy between them and we write $f \simeq g$ in this case. Fixing basepoints, it is easy to verify that the property of two maps being homotopic is transitive, symmetric and reflexive. Thus, base point preseving homotopic maps $(X, x_0) \to (Y, y_0)$ form equivalence classes, called *homotopy classes*. Write $[f]$ for the homotopy class of f.

We can now consider a type of equivalence on spaces, weaker than homeomorphism.

Definition 2.2. A *homotopy equivalence* between spaces X and Y is a map $f : X \to Y$ such that there exists an 'inverse' $g : Y \to X$ in the sense that the compositions $f \circ g$ and $g \circ f$ are homotopic to the identity:

$$f \circ g \simeq \mathrm{id}_Y \text{ and } g \circ f \simeq \mathrm{id}_X.$$

A homeomorphism is trivially a homtopy equivalence. If there is a homotopy equivalence between X and Y, then the spaces are said to be *homotopy equivalent* or to have the same *homotopy type*. Similar to above, we write $X \simeq Y$ in this case. Most of the time, we classify spaces up to homotopy equivalence, since usually it is quite hard to find homeomorphisms. In particular, homotopy (groups), homology and cohomology are all equal for spaces of the same homotopy type.

Before we are able to define homotopy groups, we need to make a few technical definitions. The *n-dimensional sphere* $S^n \subset \mathbb{R}^{n+1}$ is the set of points with norm 1 in Euclidean space:

$$S^n = \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}.$$

The *wedge sum* of two pointed spaces (X, x_0) and (Y, y_0) is the quotient space of the disjoint union of X and Y under the identification $x_0 = y_0$:

$$X \vee Y = \left(X \amalg Y\right) \Big/ \left(x_0 = y_0\right).$$

Intuitively, the wedge sum of X and Y is the union of X and Y, joined at a single point. For example $S^n \vee S^n$ is two copies of S^n touching at a single point.

The wedge sum of two basepoint preserving maps $f : (X, x_0) \to (Y, y_0)$ and $g : (X', x'_0) \to (Y', y'_0)$ is the map $f \vee g : (X, x_0) \vee (X', x'_0) \to (Y, y_0) \vee (Y', y'_0)$ defined by

$$f \vee g(x) = \begin{cases} f(x) & \text{if } x \in X, \\ g(x) & \text{if } x \in X'. \end{cases}$$

That is, $f \vee g$ maps X to Y by f and X' to Y' by g.

Let (S^n, s_0) be S^n with base point s_0. Fix an equator S^{n-1} (a great circle passing through s_0). Identify all the points on the equator. (That is, pinch the equator to a point.) Then this quotient space is homeomorphic to $S^n \vee S^n$:

$$S^n / S^{n-1} \cong S^n \vee S^n.$$

Define $c : S^n \to S^n \vee S^n$ to be the map that identifies all the points on the equator.

Now we are ready to define homotopy groups.

Definition 2.3. Let $\pi_n(X, x_0)$ be the set of homotopy classes of base point maps $f : (S^n, s_0) \to (X, x_0)$, where $n \in \mathbb{N}_0$. For $n \geq 1$, define a group operation on $\pi_n(X, x_0)$ by

$$[f] + [g] \text{ is the (homotopy class) of the composition } S^n \xrightarrow{c} S^n \vee S^n \xrightarrow{f \vee g} X.$$

Then $\pi_n(X, x_0)$ is the *n-th homotopy group* of X at base point x_0.

To make this definition valid, there are a number of things to verify:

1. $[f] + [g]$ is well defined (i.e. it doesn't depend on the choice of representation of the homotopy classes),

2. $[f] + [g]$ is associative, unital and has inverses.

These are all simple exercises.

In the case $n = 1$, the homotopy group $\pi_1(X, x_0)$ is called the fundamental group. It consists of homotopy classes of loops in X starting and finishing at x_0. The sum of two such (homotopy classes of) loops is simply the (homotopy class of the) loop which travels both loops, in the sense:

$$(f+g)(x) = \begin{cases} f(2x) & \text{if } x \in [0, \frac{1}{2}], \\ g(2x-1) & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

(We have ignored homotopy classes in the above equation to simplify notation.)

$\pi_1(\mathbb{R}^n, x_0) = 0$ for all n and x_0, since all loops in $\mathbb{R}^n$ are homotopic to the constant loop. On the other hand, $\pi_1(\mathbb{R}^2 - \{0\}, x_0) \cong \mathbb{Z}$ for all n and x_0. Informally, we can justify this as follows: consider the loops ω_n that travel around S^1 n times. Then every loop is homotopic to some ω_n and $\omega_n \simeq \omega_m$ for $n \neq m$. Also, $[\omega_1]$ generates all $[\omega_n]$. In this sense, the fundamental group measures the number of holes in the space. Higher homotopy groups 'measure higher dimensional holes'.

Theorem 2.1. If x_0 and x_0' are path connected, then $\pi_n(X, x_0) \cong \pi_n(X, x_0')$.

Proof. Let $\gamma : [0,1] \to X$ be a path from $x_0 = \gamma(0)$ to $x_0' = \gamma(1)$. Given a map $f : (D^n, S^{n-1}) \to (X, x_0)$, we can construct a new map $\gamma f : (D^n, S^{n-1}) \to (X, x_0')$ by shrinking the domain of f to a smaller concentric disk and then inserting γ on each radial line from the smaller disk to the boundary S^{n-1}.

Define a change of base point transformation $\beta_\gamma : \pi_n(X, x_0) \to \pi_n(X, x_0')$ by $\beta_\gamma([f]) = [\gamma f]$. We will show that β_γ is an isomorphism.

Firstly, we need to show that β_γ is a homomorphism: $\gamma(f+g) \simeq \gamma f + \gamma g$. We define a homotopy from $\gamma f + \gamma g$ to $\gamma(f+g)$ as follows: first deform γf and γg so they are constant on the right and left halves of D^n respectively. Then $\gamma f + \gamma g$ contain a middle constant slab. We can shrink this slab down to nothing, resulting in $\gamma(f+g)$.

It is obvious that $(\gamma\eta)f \simeq \gamma(\eta f)$ and $1f \simeq f$, where 1 is the constant path. These two facts imply that β_γ is an isomorphism. □

For path connected X, we can write $\pi_n(X)$ without reference to a particular base point. We are justified in doing this by the above theorem.

Since homotopies of maps $(S^0, s_0) \to (X, x_0)$ are equivalent to paths in X, we have that $\pi_0(X, x_0)$ is the set of path connected components of X. In particular, $\pi_0(X, x_0) = 0$ implies that X is path connected. We can extend this idea to obtain a concept of 'higher dimensional connectedness' in the following way.

Definition 2.4. A space X is n *connected* if $\pi_m(X) = 0$ for all $m \leq n$.

Important building blocks of homotopy theory are Eilenberg-MacLane spaces, which have only one non-zero homotopy group.

Definition 2.5. Given an abelian group G and $n \geq 1$, an *Eilenberg-MacLane space* of type $K(G, n)$ is characterised by:

$$\pi_m(X) \cong \begin{cases} G & \text{if } n = m, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, Eilenberg-MacLane spaces are (path) connected. It is shown that Eilenberg-MacLane spaces are unique, up to a particular technical invariant, in (Hatcher, 2002). Therefore, we talk about *the* Eilenberg-MacLane space of G and n, and denote it by $K(G, n)$.

Definition 2.6. Given a map $f : (X, x_0) \to (Y, y_0)$, the *induced map* $f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is defined by:

$$[g : (S^n, s_0) \to (X, x_0)] \mapsto [f \circ g : (S^n, s_0) \to (Y, y_0)].$$

It is straightforward to check that the induced map is indeed well defined.

3 Long Exact Sequences of Homotopy Groups

Important constructions in calculations of homotopy groups are long exact sequences. They are analogous to the long exact sequence in homology and cohomology. First, we need the notion of relative homotopy groups, generalising the ideas in the previous section.

Definition 3.1. A homotopy $f_t : X \to Y$ whose restriction to some set A is independent of t is called a *homotopy relative* to A.

Basepoint preserving homotopies are always homotopies relative to the base point x_0. Write maps $X \to Y$ that send $A \subset X$ and $A' \subset A$ to $B \subset Y$ and $B' \subset C$ respectively by $(X, A, A') \to (Y, B, B')$. Note this is a generalisation of base point preserving maps $(X, x_0) \to (Y, y_0)$ where we preserve two sets rather than single points.

Definition 3.2. For $x_0 \in A \subset X$, define the *relative homotopy* group $\pi_n(X, A, x_0)$ to be the homotopy classes of maps $(D^n, S^{n-1}, s_0) \to (X, A, x_0)$, where $n > 0$.

Note that S^{n-1} is the boundary of D^n. If $A = \{x_0\}$, then maps $(D^n, S^{n-1}, s_0) \to (X, A, x_0)$ factor through maps

$$S^n \xrightarrow{\cong} D^n / S^{n-1} \to X,$$

preserving base points, so absolute homotopy groups are a special case of relative homotopy groups.

Similar to the absolute case, we define the group operation on relative homotopy groups via the map $c : D^n \to D^n \vee D^n$ collapsing $D^{n-1} \subset D^n$ to a point.

Relative homotopy groups form *long exact sequences* in the following sense.

Definition 3.3. A sequence of groups G_n and homomorphisms $\alpha_n : G_n \to G_{n-1}$

$$... \to G_n \xrightarrow{\alpha_n} G_{n-1} \xrightarrow{\alpha_{n-1}} G_{n-2} \to ... \to G_1 \xrightarrow{\alpha_1} G_0,$$

is *exact* if $\operatorname{Im} \alpha_{n+1} \subset \ker \alpha_n$ for all n.

Theorem 3.1. There is a long exact sequence of relative homotopy groups

$$... \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to ... \to \pi_0(X, x_0)$$

where i and j are the inclusions $(A, x_0) \hookrightarrow (X, x_0)$ and $(X, x_0, x_0) \hookrightarrow (X, A, x_0)$ respectively.

See (Hatcher, 2002, pp. 344) for a proof. The map ∂, called the *boundary map* is constructed explicitly in the proof. It is a homomorphism only when $n > 1$ but exactness still makes sense if we take the kernel of a map to be those elements mapping ot the homotopy class of the constant map.

4 Suspension

Before we are able to define the stable homotopy groups of a space, we need one more geometrical construction, the suspension of a space.

Definition 4.1. The *suspension* SX of a space X is the quotient space:

$$SX = \left([0,1] \times X\right) \Big/ \left((0,x) = (0,x'), (1,x) = (1,x') \text{ for all } x \in X\right).$$

The *suspension* Sf of a map $f : X \to Y$ is:

$$\begin{array}{rccc} Sf: & SX & \to & SY \\ & (i,x) & \mapsto & (i, f(x)) \end{array}$$

The suspension of a space X can be thought of as follows: consider the cylinder space $X \times [0,1]$, then pinch the top and the bottom of the cylinder to a point. The result is two cones, one of which is inverted, connected at their bases. An example is the suspension of the unit disk $D^2 = \{x \in \mathbb{R}^2 : ||x|| \leq 1\}$. SD^2 is homeomorphic to D^3. In fact, this holds more generally: $SD^n \cong D^{n+1}$. Also, $SS^n \cong S^{n+1}$. The suspension map $Sf : SX \to SY$ simply sends each slice of X in the cylinder $X \times [0,1]$ to $f(X)$ in the corresponding slice of Y in $Y \times [0,1]$.

In the language of categories, the suspension is a functor from the category of topological spaces to itself. This will become important in section 6.

Unfortunately, we run into problems with this definition when we consider pointed spaces. The suspension takes a basepoint $x_0 \in X$ to $\{x_0\} \times [0,1]$ in SX. Any of the points on this line are possible candidates for the base point of SX. It turns out that none of these choices are compatible. Instead, we define a new construction, the *reduced suspensions*:

Definition 4.2. The *reduced suspension* ΣX of a pointed space (X, x_0) is the quotient space of SX where we identify all the points in $\{x_0\} \times [0,1]$:

$$\Sigma X = SX \Big/ \left(\{x_0\} \times [0,1]\right).$$

ΣX has the obvious basepoint: the equivalence class $[(0, x_0)]$ of the quotient.

ΣX is homotopy equivalent to SX (Hatcher, 2002, pp. 12). (Collapsing $\{x_0\} \times [0,1]$ actually gives this homotopy equivalence.) Analogously to above, we can define the reduced suspension of maps and thus construct a functor from the category of pointed spaces to itself.

Definition 4.3. The *suspension map* $\pi_n(X) \to \pi_{n+1}(\Sigma X)$ is given by:

$$[f : (S^n, s_0) \to (X, x_0)] \to [Sf : (S^{n+1}, s_0) \to (SX, (0, x_0)).$$

5 Freudenthal's Suspension Theorem and Stable Homotopy

We are now ready to present the theorem that underpins stable homotopy theory. It states that under mild conditions, the suspension map $\pi_{n+i}(S^i X) \to \pi_{n+i+1}(S^{i+1} X)$ is an isomorphism for $i >> 0$.

Theorem 5.1 (Freudenthal's Suspension Theorem)**.** The suspension map $\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$ is an isomorphism for $i < 2n-1$ and a surjection for $i = 2n-1$, whenever X is an $(n-1)$-connected CW complex.

Remark 5.1. A CW complex is geometric construction used throughout algebraic topology. It has been proven useful over time due to its generality (almost all spaces you can think of can be constructed as CW complexes and all spaces are 'nearly' CW complexes, in a sense that can be made precise) and nice behaviour (it avoids the pathological analysis examples). See (Lundell and Weingram, 1970).

of theorem 5.1. Consider SX as the union of two cones C_-X and C_+X, with $C_-X \cap C_+X = X$. The suspension map is equal to

$$\pi_i(X) \cong \pi_{i+1}(C_+X, X) \to \pi_{I+1}(SX, C_-X) \cong \pi_{i+1}(SX).$$

The two isomorphisms are given by long exact sequences of pairs. The middle map is induced by inclusion.

We need the following technical result given in (Hatcher, 2002, pp. 360): *Let X be a CW complex decomposed as the union of subcomplexes A and B with nonempty connected intersection $C = A \cap B$. If (A, C) is m-connected and (B, C) is n-connected, where $m, n \geq 0$, then the map $\pi_i(A, C) \to \pi_i(X, B)$ induced by inclusion is an isomorphism for $i < m + n$ and a surjection for $i = m + n$.*

Using long exact sequences of pairs, we see that (CX, X) is n-connected if X is $(n-1)$-connected, which gives us our result. □

A basic, yet fundamental result of Freudenthal's suspension theorem is that $\pi_n(S^n) \cong \mathbb{Z}$ for all $n \geq 1$. Using the theorem, $\pi_n(S^n) \to \pi_{n+1}(S^{n+1})$ is an isomorphism for $n \geq 2$. Using the Hopf bundle $S^1 \to S^3 \to S^2$, one finds $\pi_1(S^1) \cong \pi_2(S^2)$ (Hatcher, 2002, pp. 377). Basic theory on the fundamental groups tells us that $\pi_1(S^1) \cong \mathbb{Z}$, which implies $\pi_n(S^n) \cong \mathbb{Z}$.

Consider taking iterated suspensions of an n-connected CW complex X and consider the resulting sequence oh homotopy groups:

$$\pi_i(X) \to \pi_{i+1}(SX) \to \pi_{i+2}(S^2X) \to ...$$

Theorem 5.1 tells us that $\pi_i(X) \to \pi_{i+1}(SX)$ is an isomorphism for $i \leq n$. So SX is $(n+1)$-connected. Repeating this process, we see that in the sequence above, the maps are eventually all isomorphisms. (Note that this happens after a finite number of steps!) We say that the suspension map stabilises and these isomorphic groups are given a special name.

Definition 5.1. The ith *stable homotopy group* $pi_i^s(X)$ is the above result of iterated suspensions. In other words,

$$\pi_i^s(X) = \operatorname*{colim}_n \pi_{i+n}(S^nX),$$

where colim is the colimit of the above sequence of iterated suspensions.

Remark 5.2. The colimit is a generalised notion of a limit in category theory. See (Leinster, 2014) for more information. In the category of abelain groups, the definition simplifies to the following: the colimit $\operatorname*{colim}_n G_n$ of a sequence of homomorphisms of abelian groups $G_1 \xrightarrow{\alpha_1} G_2 \xrightarrow{\alpha_2} G_3 \to ...$ is the direct sum $\oplus_n G_n$ mod the subgroup consisting of elements of the form $(g_1, g_2 - \alpha_1(g_1), g_3 - \alpha_2(g_2), ...)$.

A special case of stable homotopy groups are the *stable i-stem* $\pi_i^s(S^0)$. It is easy to see, using Freudenthal's suspension theorem, that $\pi_i^s(S^0) \cong \pi_{i+n}(S^n)$ for $n > i+1$. It is proved in (Hatcher, 2004) that $\pi_i^s(S^0)$ is finite for all $i > 0$. Given the fundamental nature of the stable i-stems, it is perhaps surprising that we do not know $\pi_i^s(S^0)$ for i greater than approximately 60 (Hatcher, 2002, pp. 384). Below is a table of the stable i-stems for $i \leq 12$ (Toda, 1962):

i	$\pi_i^s(S^0)$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_{24}$
4	0
5	0
6	$\mathbb{Z}_2$
7	$\mathbb{Z}_{240}$
8	$\mathbb{Z}_2 \times \mathbb{Z}_2$
9	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
10	$\mathbb{Z}_6$
11	$\mathbb{Z}_{504}$
12	0

6 Spectra

A spectrum is a sequence of spaces X_n with maps connecting X_n and X_{n+1}. It turns out that spectrum are similar to spaces, but better behaved. They are useful tools to study stable homotopy theory. We will define two flavours of spectrum and then discuss how all spectrum are 'roughly' equivalent.

Definition 6.1. The Σ-spectrum consists of pointed spaces X_n, $n \geq 0$, with base point preserving maps $\Sigma X_n \to X_{n+1}$.

There are two important examples of Σ-spectrum. The obvious one is the *suspension spectrum* where $X_n = \Sigma^n X$ for some pointed space X with $\Sigma X_n \to X_{n+1}$ the identity map. The second example is *Eilenberg-MacLane spectrum* for abelian groups G, which we will come back to in section 7.

Before we can define the second type of spectrum, we need to develop some theory.

Definition 6.2. The *loopspace* ΩX of a pointed space X is the space of loops $f : (S^1, s_0) \to (X, x_0)$ at x_0. The basepoint of ΩX is taken to be the constant loop. The topology on ΩX is given by considering ΩX as a subspace of the space of all maps $I \to X$ with the compact-open topology.

Definition 6.3. The map $f : X \to Y$ is a *weak homotopy equivalence* if it induces isomorphisms between homotopy groups:

$$f_* : \pi_n(X, x_0) \xrightarrow{\cong} \pi_n(Y, f(x_0)) \text{ for all } n \geq 0 \text{ and all } x_0 \in X.$$

Definition 6.4. An Ω-*spectrum* is a sequence of CW complexes X_n with weak homotopy equivalences $X_n \to \Omega X_{n+1}$.

There are a number of other definitions of spectra. See (Malkiewich, 2014) for examples. Recall that we are interested in spectra to help us understand stable homotopy category. The important properties are that each flavour of spectra form a category and these categories in turn each define a stable homotopy category. While the categories of spectra are not all equivalent, the corresponding stable homotopy categories are almost always equivalent (Malkiewich, 2014).

We will spend the rest of this section describing these two categories. Let $\mathbf{Top}_*$ be the *category of pointed topological spaces* and $\mathbf{CW}_*$ be the *category of pointed CW complexes*, both with base point preserving maps. (Technically the morphisms between X and Y in $\mathbf{Top}_*$ are maps between X and Y's CW approximations.) Define the *homotopy category of pointed topological spaces* $\mathbf{HoTop}_*$ to have the same objects as $\mathbf{Top}_*$ but the morphisms are now homotopy classes of maps.

(Malkiewich, 2014) constructs two types of categories **Spectra** and **HoSpectra** all with the following properties:

- There are functors $\Sigma^\infty : \mathbf{CW}_* \to \mathbf{Spectra}$ and $\Sigma^\infty : \mathbf{HoTop}_* \to \mathbf{HoSpectra}$.
- There is a suspension functor $\Sigma : \mathbf{HoSpectra} \to \mathbf{HoSpectra}$, which agrees with the reduced suspension functor Σ of $\mathbf{CW}_*$, in the sense that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{CW}_* & \xrightarrow{\Sigma} & \mathbf{CW}_* \\ \downarrow{\scriptstyle \Sigma^\infty} & & \downarrow{\scriptstyle \Sigma^\infty} \\ \mathbf{HoSpectra} & \xrightarrow{\Sigma} & \mathbf{HoSpectra} \end{array}$$

  The nice thing in this case is that Σ is an equivalence of categories from **HoSpectra** to itself. This means that every object is isomorphic to the suspension of another object, behaviour which we do not have in $\mathbf{CW}_*$.

- Finally, it is possible to construct stable homotopy groups $\pi_n(X)$ in **HoSpectra**, which is what we cared about to start with. The good thing is that they live in a much more nicely behaved category than $\mathbf{CW}_*$ or $\mathbf{Top}_*$.

7 Brown's Representability Theorem

Let $\langle X, Y\rangle$ be the set of base point preserving homotopy classes of maps from X to Y. Given some special structure on X or Y, we can define a group operation on $\langle X, Y\rangle$. For example, if $X = S^n$ then we have $\langle S^n, Y\rangle = \pi_n(Y)$. More generally, replace S^n with the reduced suspension ΣX of any space X. The sum of maps f, g in this case is defined as the composition

$$\Sigma X \xrightarrow{c} \Sigma X \vee \Sigma X \xrightarrow{f \vee g} Y.$$

Inverses are given by reflecting the I co-ordinate in the suspension:

$$-f(x, i) = f(x, 1 - i).$$

It is easy to verify that this does indeed give a group structure on $\langle SX, Y\rangle$. This construction hints at how a group operation on spectrum is defined.

We can also give a group operation to $\langle X, Y\rangle$ if Y has a special structure, using the adjoint relation:

$$\langle \Sigma X, Y\rangle = \langle X, \Omega Y\rangle.$$

This says that base point preserving maps $\Sigma X \to Y$ are exactly the same as base point preserving maps $X \to \Omega Y$. The correspondence between the two types of maps is given by

$$\big(f : \Sigma X \to Y\big) \mapsto \big(x \mapsto f(\{x\} \times [0,1]) \in \Omega Y\big).$$

That is, associate f with the family of loops $f(\{x\} \times [0,1])$.

Using the adjoint relation, we can see the close connection between Σ- and $\Omega-$ spectra. Moreover, we are qualified to define the *Eilenberg MacLane spectrum for an abelian group G*. This spectrum has Eilenberg MacLane spaces $K(G,n)$ and maps $\Sigma K(G,n) \to K(G,n+1)$ given as the adjoint of a special map, called 'the CW approximation' $K(G,n) \to \Omega K(G,n+1)$.

We have the following theorem (Hatcher, 2002, pp. 397):

Theorem 7.1. If $\{K_n\}$ is an Ω-spectrum, then the functors $X \mapsto h^n(X) = \langle X, K_n\rangle$ for $n \in \mathbb{Z}$, define a reduced cohomology theory on the category of pointed CW complexes and base point preserving maps.

The converse of this statement is Brown's representability theorem (Brown, 1962):

Theorem 7.2. Every reduced cohomology theory on the category of pointed CW complexes and base point preserving maps has the form $h^n(X) = \langle X, K_n\rangle$ for some Ω-spectrum $\{K_n\}$.

This is quite amazing. It says that we can completely describe cohomology theories in terms Ω-spectrum and homotopy classes of maps.

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