Stable Homotopy Theory

James Bailie
Supervised by Dr. Vigleik Angelveit
Australian National University

Sponsors

Australian Government
Department of Education and Training

AMSI
AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

February 8, 2017

Homotopy

Definition

A *homotopy* between $f : X \to Y$ and $g : X \to Y$ is a family of functions $F_t : X \to Y$ for $t \in [0, 1]$ such that

1. $F_0 = f$,
2. $F_1 = g$,
3. $F_t(x)$ is continuous in both t and x.

f and g are said to be *homotopic* if there is a homotopy between them. This property partitions maps $X \to Y$ into equivalence classes, called *homotopy classes*.

Homotopy Groups $\pi_n(X)$

Definition

$\pi_n(X)$ is the set of homotopy classes of maps $f : S^n \to X$.
For $n \geq 1$, define a group operation on $\pi_n(X)$ by

$$f + g \text{ is the composition } S^n \xrightarrow{c} S^n \vee S^n \xrightarrow{f \vee g} X$$

$\pi_1(X)$ is called the *fundamental group*.

Suspensions

Definition

The *suspension* of X is

$$SX = (X \times [0,1])/(X \times \{0\} \cup X \times \{1\}).$$

The *suspension* of $f : X \to Y$ is a map $Sf : SX \to SY$ defined by

$$Sf([x,t]) = [f(x), t].$$

Example

The suspension of S^n is S^{n+1}.

Definition

The *suspension map* $\pi_n(X) \to \pi_{n+1}(SX)$ is given by

$$[f : S^n \to X] \to [Sf : S^{n+1} \to SX].$$

Freudenthal Suspension Theorem and Stable Homotopy Groups

Theorem

Under mild conditions, the suspension map $\pi_{n+i}(S^i X) \to \pi_{n+i+1}(S^{i+1}X)$ *is an isomorphism for* $i >> 0$*.*

Definition

The *nth stable homotopy group* of X is

$$\pi_n^s(X) = \operatorname*{colim}_i \pi_{n+i}(S^i X).$$

Thank you. Questions?